ANALYTIC ELEMENTS FOR TRANSIENT GROUNDWATER FLOW

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA

BY
WILLEM JAN ZAADNOORDIJK

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

1988
revised 2003

abstract

In this thesis an extension of the analytic element method to transient regional flow is presented. The elements are solutions to the heat equation in two spatial dimensions with a sink term.

Strack(1988) developed the analytic element method for the computer modeling of steady regional groundwater flow. The method is based on the superposition of solutions to the governing differential equation.

Most of the transient elements that are derived in this thesis, are based on the solution for an instantaneous point sink or point doublet (Carslaw and Jaeger, 1986). The elements are transient point-, line- and area-sinks and line-doublets. Some are new solutions to the heat equation in an infinite domain.

The elements based on the instantaneous well have no influence at infinity. The consequence of this is that a model containing these elements has to fulfill strict conditions, in order to converge to a final steady state in the entire area modeled. Two functions are presented that make it possible to waive these conditions. These are given the names of transient far-field functions in this thesis.

In modeling a practical situation, the behavior of the model without the transient far-field functions will be correct in the area of interest, only if all aquifer features are included that have a significant influence on the flow in the area of interest.

Some of the elements have been incorporated in a computer program. Results obtained with the program compared well with exact solutions. An application of the computer program to an example of regional groundwater flow is presented at the end of this thesis.

acknowledgment

Dr. O.D.L. Strack has been my advisor during my PhD studies. The Legislative Commission on Minnesota Resources provided funding to carry out the research on which this thesis is based.

The original version of the thesis had been published in 1988. This revised version has been produced in 2003. The revisions have been prompted by an error in the transient line element found by Barnes & Strack (2003). Many of the equations in Appendix E have been corrected as well as some equations in Chapter 4 and in Appendix A. The Fortran code in Appendix A has been revised also. The error involved only the call of the function. The revisions of the source code consist of conversion to double precision, cleanup of the code and a better implementation of the integral that is also due to Barnes & Strack (2003).

I hope that this revised version will be useful to people who want to use the analytic element method for transient modeling, in particular using the object oriented analytic element program Tim (see http://www.engr.uga.edu/ mbakker/tim.html).

In the years that have passed since I got my Ph.D. I have come to realize how much I have learned from Otto Strack and I am very greatful to him for the valuable knowledge and experience.

content

1. introduction

The subject area of this thesis is modeling transient regional groundwater flow. The goal of the modeling can be for instance to predict levels of groundwater in a future situation or to examine the spreading of contaminants with the water.

The following simplifying assumptions have been made:

- Darcy's law is valid and the hydraulic conductivity is isotropic
- the properties of the groundwater are constant
- the properties of the aquifer are constant
- the base of the aquifer is horizontal
- the flow is shallow
- the storage and release of water is linear, occurs instantaneously and is fully reversible
- the variations of the saturated thickness in unconfined flow are small compared to the saturated thickness itself.

The analytic element method will be used. Strack(1988) developed the analytic element method for the modeling of steady regional groundwater flow. The method is based on the superposition of solutions to the differential equation. Exact solutions to a variety of boundary value problems have been used for a long time in simple cases, for example the flow to a well in an infinite aquifer. There are various functions available in the literature for the transient response of the water table to a single well that can be used to evaluate pumping tests.

Superposition is used to make it possible to model more complex groundwater systems efficiently. The functions that are superimposed are solutions to the governing differential equation with a behavior that is suitable to represent an aquifer feature. They are called analytic elements. They have coefficients that are a priori unknown. The unknown coefficients are determined such that the boundary conditions are met exactly at selected points. Once the coefficients have been determined, the level and the velocities of the groundwater are known as a function of position.

In this thesis an extension of the analytic element method is presented to transient regional flow. Most of the time dependent elements are based on the solution for an instantaneous sink (Carslaw and Jaeger, 1986).

Some of the transient elements are available in the literature (Carslaw and Jaeger, 1986, Theis, 1935, Glover, 1974 and Litkouhi and Beck, 1982). Others are new solutions to the heat equation in an infinite domain either without or with a sink-term.

The work on which this thesis is based has been motivated by a project funded by the Legislative Commission on Minnesota Resources of the state of Minnesota. The objective was to be able to model seasonal variations in groundwater flow and the transition from one steady state to a new one. By including seasonal variations into a model, more detailed information about contaminant spreading can be obtained than is possible with a steady state model of an average situation. The transition to a new situation has to be modeled with a transient program to be able to tell how long it will take until the groundwater has adjusted to the new situation.

The setup of this thesis is as follows. In chapter 2 the terms are defined that are used in the mathematical description of transient groundwater flow and the governing equations are given. The analytic element method for steady regional groundwater flow is described in chapter 3, with a description of the important idea of the "far-field" in a model. The transient elements are presented in chapter 4. In chapter 5 the behavior of elements based on the instantaneous sink is examined for large values of the time. Two functions are given which make it possible to use these elements in any model for large values of the time. In chapter 6 it is explained how the derived elements can be used in a model. The model was incorporated in two computer programs. In one program the head is constant initially, while an initial steady state simulated with a computer model for steady flow can be used in the other program. The model was validated in chapter 7 for some cases, for which exact solutions are available using the former program. The latter program was used to model a more realistic situation in chapter 8. In chapter 9 some concluding remarks are given.

Six appendices are included: in appendices A through D functions that recur in several elements are discussed, together with limits and derivatives of those functions. In appendix E the derivation is given of a line-sink of arbitrary degree, which is presented in chapter 4. Appendix F contains a list of symbols.

2. definitions and equations

In this chapter a two dimensional mathematical description of time dependent groundwater flow is presented. The description has been taken from Strack(1988) and Bear and Verruijt(1987), and can be found in more detail in these references.

Groundwater flows in geologic formations that are called aquifers. In regional groundwater flow their horizontal extent usually is much larger than their thickness, so that the flow is shallow. There are two kinds of aquifers, confined and unconfined aquifers. A confined aquifer is fully saturated with water. The water in an unconfined aquifer has a free surface. Water can evaporate into the air or leave the aquifer through the base or the top. The water leaving the aquifer is called the extraction.

A cartesian coordinate system is used. The x and y axes are horizontal. The vertical coordinate z is zero at the base of the aquifer and positive above it.

The driving force for flow is the gradient of the piezometric head. The piezometric head φ is defined as

$$\varphi = \varphi(x, y, z) = \frac{p(x, y, z)}{\rho g} + z \tag{2.1}$$

where p is the pressure, ρ the density of the water and g the acceleration of gravity.

The flux of the fluid is referred to as the magnitude of the specific discharge vector q_j. The index refers to the direction of the flow. The specific discharge is equal to the amount of water that passes through a plane normal to the direction per unit area per unit time. Darcy's law relates the specific discharge to the hydraulic gradient

$$q_j = -k\frac{\partial \varphi}{\partial j} \qquad j = x, y, z \tag{2.2}$$

The proportionality constant k is called the hydraulic conductivity.

The components of the seepage velocity of the water v_j are equal to

$$v_j = \frac{q_j}{\nu} \qquad j = x, y, z \tag{2.3}$$

where ν is the porosity, the ratio between the volume of the pores and the total volume of a block of porous material.

The Dupuit-Forchheimer assumption is valid for shallow flow. The assumption states that the head does not vary in vertical direction

$$\varphi = \varphi(x, y) \tag{2.4}$$

The horizontal components of the specific discharge then also are independent of the vertical coordinate, as can be seen from (2.2)

$$q_j = q_j(x, y) \qquad j = x, y \tag{2.5}$$

The discharge vector Q_j is defined as the vertically integrated specific discharge

$$\begin{aligned} Q_j &= \int_h q_j \, dz \\ &= h q_j \qquad\qquad j = x, y \end{aligned} \tag{2.6}$$

where h is the saturated thickness of the aquifer. It is equal to

$$h = \begin{cases} H & \text{confined} \\ \varphi & \text{unconfined} \end{cases} \tag{2.7}$$

where H is the thickness of the confined aquifer.

A continuity condition can be stated in terms of the discharge vector Q_j

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + E + S\frac{\partial \varphi}{\partial t} = 0 \tag{2.8}$$

where E is the extraction of the aquifer and S is the storativity. The storativity of the aquifer has different values for confined and for unconfined flow. (Bear and Verruijt, 1987, equation 4.1.1, 4.1.2, 4.1.3).

In the mathematical description it is convenient to use a discharge potential Φ as main dependent variable. The definition of a discharge potential is such that its gradient is equal to minus the discharge vector

$$\frac{\partial}{\partial j}\Phi = -Q_j \qquad j = x, y \tag{2.9}$$

The relations between the discharge potential and the piezometric head are different for confined and for unconfined flow, since the saturated thickness in (2.6) is different in each case.

If the discharge potential for confined flow is given by

$$\Phi = kH\varphi - \frac{1}{2}kH^2 \qquad \varphi \geq H \tag{2.10}$$

then the equation (2.9) and the continuity condition (2.8) can be combined into the differential equation

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = E + S\frac{1}{kH}\frac{\partial}{\partial t}\Phi \qquad \varphi \geq H \tag{2.11}$$

which can be written as

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = E + \frac{1}{\alpha}\frac{\partial}{\partial t}\Phi \qquad \varphi \geq H \tag{2.12}$$

when an aquifer diffusivity α (compare Carslaw and Jaeger, 1986) is defined as

$$\alpha = \frac{kH}{S} \qquad \varphi \geq H \tag{2.13}$$

The relation between the discharge potential and the piezometric head for unconfined flow is

$$\Phi = \frac{1}{2}k\varphi^2 \qquad \varphi < H \tag{2.14}$$

A linearization is carried out by setting

$$\varphi = \bar{h} + \varepsilon \tag{2.15}$$

where the variations in the saturated thickness ε are assumed to be much smaller than the average value $\bar{h}$

$$\varepsilon << \bar{h} \tag{2.16}$$

so that

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}k\varphi^2\right] = k\varphi\frac{\partial \varphi}{\partial t} = k(\bar{h} + \varepsilon)\frac{\partial \varphi}{\partial t} \simeq k\bar{h}\frac{\partial \varphi}{\partial t} \tag{2.17}$$

Combination of the equations (2.9), (2.8) and (2.17) gives the partial differential equation

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = E + \frac{S}{k\bar{h}}\frac{\partial}{\partial t}\Phi \tag{2.18}$$

which can also be written in the form (2.12), if the aquifer diffusivity for unconfined flow is set equal to

$$\alpha = \frac{k\bar{h}}{S} \qquad \varphi < H \tag{2.19}$$

Thus the partial differential equation (2.12)

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = E + \frac{1}{\alpha}\frac{\partial}{\partial t}\Phi \tag{2.20}$$

describes both confined and unconfined flow. However, the relations between the discharge potential Φ and the piezometric head and the expressions for the aquifer diffusivity α are different in each case (for confined flow (2.10) and (2.13) and for unconfined flow (2.14) and (2.19) respectively).

The differential equation is a parabolic partial differential equation. If the sink term is equal to zero it reduces to the heat equation (Weinberger, 1965).

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = \frac{1}{\alpha}\frac{\partial}{\partial t}\Phi \tag{2.21}$$

If the flow is steady, the partial differential equations (2.20) and (2.21) can be reduced to respectively Poisson's equation

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = E \tag{2.22}$$

where the extraction E does not vary with time, and Laplace's equation

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = 0 \tag{2.23}$$

The partial differential operator in the latter four partial differential equations is linear, so that the superposition principle can be used, provided that only one value of the diffusivity α is used. The principle can be applied to two solutions of one equation or to solutions of different equations.

Let Φ_1 and Φ_2 be two solutions of the heat equation with a sink-term (2.20). Then the sum $\Phi_1 + \Phi_2$ also is a solution of the differential equation (2.20) with the same value of the aquifer diffusivity α. If the extraction of Φ_1 is equal to E_1 and Φ_2 gives E_2 then the sum $\Phi_1 + \Phi_2$ gives an extraction that is equal to $E_1 + E_2$.

In general the total potential which is a solution of (2.20) can be divided into two parts, Φ and $\overset{\text{steady}}{\Phi}$. The first part is transient. It fulfills the heat equation with a transient sink term. Φ will be used for the potentials of the transient elements, which are derived in this thesis. The second part, $\overset{\text{steady}}{\Phi}$, fulfills Poisson's equation.

Similarly the potential can be split into a part without extraction and a part with extraction. Therefore, the steady potential is the sum of solutions of Poisson's equation (2.22) and of Laplace's equation (2.23). The transient potential is the sum of solutions to the heat equation (2.21) and the heat equation with a transient sink term (2.20).

3. analytic element method for steady flow

In this chapter an overview is given of modeling steady regional groundwater flow using the analytic element method. The method is covered in detail in Strack(1988); only the principles will be given here, along with a brief overview of some elements. The steady analytic element method is discussed here for two reasons. The initial state of a transient problem is assumed to be steady in this thesis, and steady elements are limiting cases for corresponding transient elements at large values of time.

The analytic element method is based on the superposition of analytic functions. Each function represents a particular feature of the aquifer. These functions are referred to as analytic elements; there exist analytic elements for modeling such features as uniform flow, wells, creeks, rivers and inhomogeneities in the aquifer properties.

Each function is selected such that it is suitable to simulate the effect of the aquifer feature in question. At least one degree of freedom is associated with each element. The degrees of freedom are represented as parameters in the analytic element.

Steady analytic elements are solutions to either Laplace's equation (2.23) or Poisson's equation (2.22). The domain is the entire plane except isolated points, line-segments or internal regions. Boundary conditions at an element are specific for the particular element. Depending on the type of element, there are different conditions at infinity that are satisfied.

elements

All analytic elements that are presented here are taken from Strack(1988). In this reference a complex potential is used in many cases. The complex potential is a complex function of the complex location $x + \mathrm{i}y$. The discharge potential is equal to the real part of the complex potential

$$\Phi = \Re\{\Omega(x + \mathrm{i}y)\} \tag{3.1}$$

uniform evapotranspiration.

The potential for uniform evapotranspiration gives a constant extraction E_1 throughout the entire plane. It can also be referred to as evapotranspiration of order one. The equipotentials are ellipses (see figure 3.1).

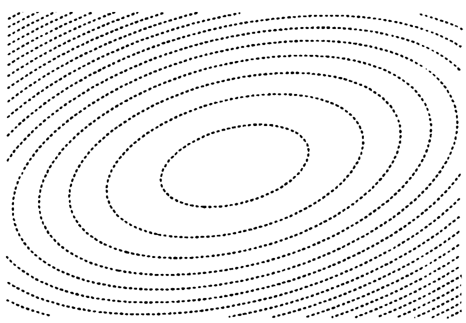

figure 3.1. *Equipotentials for evapotranspiration of order one.*

uniform flow.

The discharge due to uniform flow has the same magnitude and the same direction at every point of the plane. The flow is one dimensional (see figure 3.2).

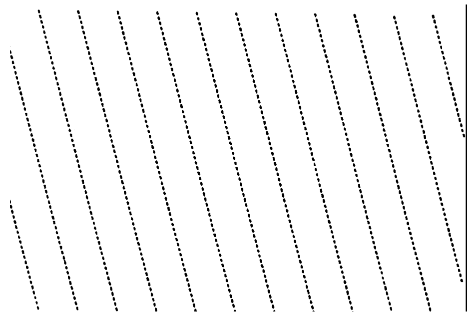

figure 3.2. *Equipotentials for uniform flow.*

constant.

The constant has one value in the entire plane.

well.

A well removes an amount Q of water per time from the aquifer. The boundary condition at the element is given by

$$\lim_{r\to 0} -2\pi r Q_r = Q \tag{3.2}$$

where r is a radial coordinate centered at the well and Q is the discharge of the well.

The complex potential for a steady well is equal to

$$\overset{\text{steady}}{\underset{w}{\Omega}} = \frac{Q}{2\pi}\ln(\frac{x+\mathrm{i}y}{R}) \tag{3.3}$$

where the subscript w stands for well and R is a length. The length R is the radius of the circle around the well where the value of the discharge potential for the well is equal to zero. The discharge potential is equal to

$$\overset{\text{steady}}{\underset{w}{\Phi}} = \frac{Q}{4\pi}\ln(\frac{x^2+y^2}{R^2}) \tag{3.4}$$

The discharge in x-direction

$$\overset{\text{steady}}{\underset{w}{Q_x}} = -\frac{Q}{2\pi}\frac{x}{x^2+y^2} \tag{3.5}$$

and the discharge in y-direction

$$\overset{\text{steady}}{\underset{w}{Q_x}} = -\frac{Q}{2\pi}\frac{y}{x^2+y^2} \tag{3.6}$$

and the discharge in the radial direction r

$$\overset{\text{steady}}{\underset{w}{Q_r}} = -\frac{Q}{2\pi}\frac{1}{r} \tag{3.7}$$

The potential (3.4) is unbounded at infinity. The discharges (3.5) through (3.7) vanish at infinity but the total flow does not approach zero

$$\lim_{r\to\infty} 2\pi r Q_r = -Q \tag{3.8}$$

doublet.

A doublet is obtained by taking the limit of two wells with opposite discharges approaching each other while the distance times discharge remains constant. A pattern of equipotentials is given in figure 3.3

The orientation of the doublet is indicated by an arrow in the figure. This is the direction from which the well with positive strength approached the well of negative strength. It also is the direction in which the water approaches the dipole on one side and flows away on the other side.

A doublet does not have a net discharge, since the two wells have equal, but opposite discharges.. The potential is equal to zero at infinity.

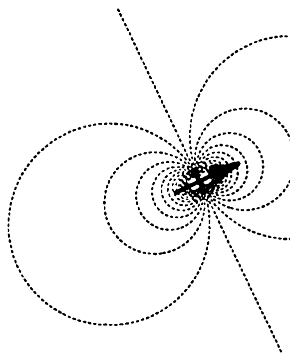

figure 3.3. *Equipotentials for a doublet*

line-elements.

A local coordinate system ξ, η is used for line-elements (see figure 3.4). The element lies on the ξ axis, so that the η axis is normal to the element. The end-points of the element are $(\xi_1, 0)$ and $(\xi_2, 0)$.

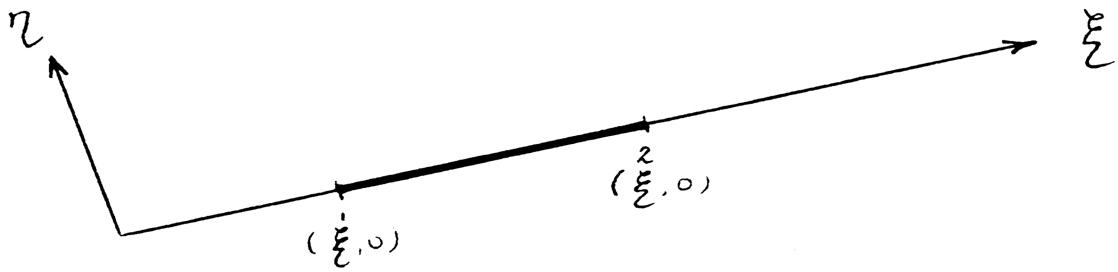

figure 3.4. *Local coordinates for line-elements.*

Strack(1988) expresses the potentials for line-elements in terms of a a complex variable $Z = X + \mathrm{i}Y$ (see figure 3.5). The relation between Z and the local coordinates ξ, η is given by

$$\begin{aligned} Z =& \frac{\xi + \mathrm{i}\eta - \frac{\xi_1+\xi_2}{2}}{\frac{\xi_2-\xi_1}{2}} \\ =& \frac{2(\frac{\xi}{R} + \mathrm{i}\frac{\eta}{R}) - \frac{\xi_1}{R} - \frac{\xi_2}{R}}{\frac{\xi_2}{R} - \frac{\xi_1}{R}} \end{aligned} \tag{3.9}$$

where R is a length.

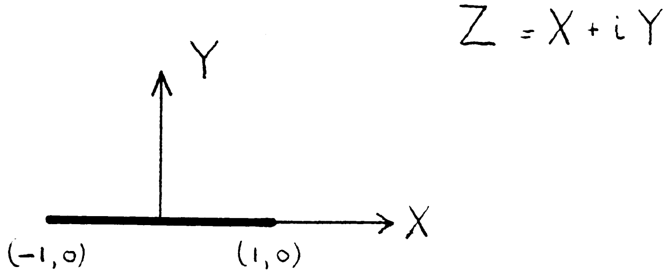

figure 3.5. *Line-element in complex variable Z*

line-sink.

A line-sink removes a discharge σ from the aquifer along the element. This means that there is a jump in the normal discharge across the element with a magnitude σ. The boundary condition along the element is

$$\lim_{\eta\downarrow 0} Q_\eta - \lim_{\eta\uparrow 0} Q_\eta = -\sigma \qquad \xi_1 < \xi < \xi_2 \tag{3.10}$$

The behavior at infinity of the potential (3.13) is just like that of the potential for a well (3.4).

$$\overset{\text{steady}}{\underset{ls}{\Phi}} = \overset{\text{steady}}{\underset{w}{\Phi}}\Big|_{Q=\sigma|\xi_2-\xi_1|} \qquad (\sqrt{x^2+y^2}\to\infty) \tag{3.11}$$

A line-sink can be obtained by integrating a well along a line-segment. If the jump in the normal discharge σ is constant along the element, the line-sink is said to be of order one. The complex potential is equal to

$$\begin{aligned}\overset{\text{steady}}{\underset{ls}{\Omega}} =&\sigma\frac{\xi_2-\xi_1}{4\pi}[(Z+1)\ln(Z+1)\\ &-(Z-1)\ln(Z-1)+\ln(\frac{\xi_2-\xi_1}{2R})-2]\\ =&\sigma[-\frac{1}{2\pi}\{(\xi-\xi_2+i\eta)\ln(\frac{\xi-\xi_2)+i\eta}{R})-(\xi-\xi_1+i\eta)\ln(\frac{\xi-\xi_1)+i\eta}{R})\}\\ &-\frac{\xi_2-\xi_1}{2\pi}]\end{aligned} \tag{3.12}$$

where the subscript ls stands for line-sink and the complex variable Z is given in (3.9).

The discharge potential is given by

$$\begin{aligned}\overset{\text{steady}}{\underset{ls}{\Phi}} =&\sigma[-\frac{1}{4\pi}\{(\xi-\xi_2)\ln(\frac{(\xi-\xi_2)^2+\eta^2}{R^2})-(\xi-\xi_1)\ln(\frac{(\xi-\xi_1)^2+\eta^2}{R^2})\}\\ &+\frac{\eta}{2\pi}\{\arctan(\frac{\eta}{\xi-\xi_2})-\arctan(\frac{\eta}{\xi-\xi_2})\}\\ &-\frac{\xi_2-\xi_1}{2\pi}]\end{aligned} \tag{3.13}$$

The discharge parallel to the element is the discharge in ξ-direction

$$\overset{\text{steady}}{\underset{ls}{Q_\xi}} = \frac{\sigma}{4\pi}\{\ln(\frac{(\xi-\xi_2)^2+\eta^2}{R^2})-\ln(\frac{(\xi-\xi_1)^2+\eta^2}{R^2})\} \tag{3.14}$$

The discharge in η-direction of a first order line-sink

$$\overset{\text{steady}}{\underset{ls}{Q_\eta}} = \frac{\sigma}{2\pi}\{\arctan\frac{\xi-\xi_2}{\eta} - \arctan\frac{\xi-\xi_1}{\eta}\} \tag{3.15}$$

line-doublet.

A line-doublet is a line-segment along which the potential is discontinuous. The jump in the potential is equal to the strength of the line-doublet λ.

The boundary condition along the element is

$$\lim_{\eta\downarrow 0}\Phi - \lim_{\eta\uparrow 0}\Phi = \lambda \qquad \xi_1 < \xi < \xi_2 \tag{3.16}$$

At infinity the potential (3.19) is regular and equal to zero just like the potential of the doublet

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty} \Phi = 0 \tag{3.17}$$

The potential for a line-doublet can be obtained by integrating a the potential for a doublet doublet along aline-segment. The orientation of the doublet is normal to line along which is integrated. The jump in the potential across the element varies linearly along a line-doublet of order one. It is denoted by the strength $\lambda_c+\xi\lambda_l$, where the subscripts c and l indicate the constant and linear part, respectively. The complex potential is equal to

$$\overset{\text{steady}}{\underset{db}{\Omega}} = \frac{\lambda_c+\lambda_l\frac{Z(\xi_2-\xi_1)+\xi_1+\xi_2}{2}}{2\pi\mathrm{i}}\ln(\frac{Z-1}{Z+1}) + \frac{\lambda_l}{2\pi\mathrm{i}}(\xi_2-\xi_1) \tag{3.18}$$

where the subscript db stands for line-doublet and the complex variable Z is given in (3.9).

The discharge potential is equal to

$$\begin{aligned}\overset{\text{steady}}{\underset{db}{\Phi}} &= (\lambda_c+\xi\lambda_l)[\frac{1}{2\pi}\{\arctan\frac{\xi-\xi_2}{\eta} - \arctan\frac{\xi-\xi_1}{\eta}\}] \\ &+\lambda_l[\frac{\eta}{4\pi}\{\ln(\frac{(\xi-\xi_2)^2+\eta^2}{R^2}) - \ln(\frac{(\xi-\xi_1)^2+\eta^2}{R^2})\}]\end{aligned} \tag{3.19}$$

which gives a discharge in the ξ-direction

$$\begin{aligned}\overset{\text{steady}}{\underset{db}{Q_\xi}} &= (\lambda_c+\xi\lambda_l)[\frac{\eta}{2\pi}\{\frac{1}{(\xi-\xi_2)^2+\eta^2} - \frac{1}{(\xi-\xi_1)^2+\eta^2}\} \\ &+\lambda_l[-\frac{\eta}{2\pi}\{\frac{\xi-\xi_2}{(\xi-\xi_2)^2+\eta^2} - \frac{\xi-\xi_1}{(\xi-\xi_1)^2+\eta^2}\} \\ &-\frac{1}{2\pi}\{\arctan(\frac{\eta}{\xi-\xi_2}) - \arctan(\frac{\eta}{\xi-\xi_1})\}]\end{aligned} \tag{3.20}$$

where the range of the arctangents is from $-\pi$ to π. The discharge in the η-direction is equal to

$$\begin{aligned}\overset{\text{steady}}{\underset{db}{Q_\eta}} &= (\lambda_c+\xi\lambda_l)[-\frac{1}{2\pi}\{\frac{\xi-\xi_2}{(\xi-\xi_2)^2+\eta^2} - \frac{\xi-\xi_1}{(\xi-\xi_1)^2+\eta^2}\} \\ &+\lambda_l[-\frac{\eta^2}{2\pi}\{\frac{1}{(\xi-\xi_2)^2+\eta^2} - \frac{1}{(\xi-\xi_1)^2+\eta^2}\} \\ &-\frac{1}{4\pi}\{\ln(\frac{(\xi-\xi_2)^2+\eta^2}{R^2}) - \ln(\frac{(\xi-\xi_1)^2+\eta^2}{R^2})\}]\end{aligned} \tag{3.21}$$

area-sink.

An area-sink removes water from the aquifer inside the element. The extraction is constant for an area sink of order one. The potential approaches the potential for a well of the same discharge far away. Equipotentials for an area-sink are shown in figure 3.6.

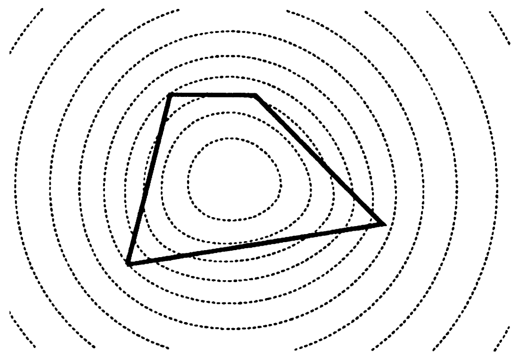

figure 3.6. *Equipotentials for an area-sink*

modeling

A brief description of modeling a flow problem with steady analytic elements will be given. Modeling with the steady analytic element method is discussed extentively in Strack(1988) and Strack(1986). Boundary conditions are applied at selected points (called control points) of the boundaries of the aquifer features: these conditions give rise to one equation per control point. There is one more parameter in the solution. It does not correspond to an aquifer feature. This the constant, and the associated equation is the reference condition: a given value of the head (the reference head) at a specified location (the reference point). The total number of control points is equal to the total number of parameters in the solution; the system of equations is solved for the unknown parameters.

Once all parameters are determined, an approximate analytic solution to the problem is known. The solution is approximate because the boundary conditions are not met at every point of the boundary but only at the selected control points. Although approximate, this solution is truly analytic: Piezometric heads can be computed at any point in the aquifer without the need for interpolation, and velocities can be obtained by analytic differentiation. The latter property is of particular advantage in problems of contaminant transport as numerical dispersion is of no concern.

In modeling a regional flow problem with the Analytic Element Method the aquifer features in the area of interest are represented in great detail by analytic elements. Around the area of interest the near-field is chosen. The term near-field is used for the area in which individual aquifer features are represented explicitly by analytic elements. The representation of the aquifer features in the near-field becomes gradually coarser as the distance to the area of interest increases. The farther away, the less do details influence the groundwater in the area of interest.

The necessary size of the near-field is determined interactively. An estimate is made of the size and elements are entered inside the near-field. The size of the near-field is then increased in steps until the effect of the additional elements on the area of interest becomes negligible (Strack, 1986).

The area outside the near-field is called the far-field. The far-field functions are the constant, uniform flow and uniform extraction and the discharge at infinity. If the near-field is not extended far enough, then it becomes important to choose the proper far-field. The far-field then has to represent the important aquifer features that have been left out.

So a complete model of a flow problem consists of a near-field with the area of interest in its center, where the aquifer features are modeled explicitly by elements, and a far-field, where the aquifer features are implicitly represented by the far-field functions. The far-field is not important for the modeling of the area of interest if the near-field is sufficiently large.

The model only gives reliable information in the area of interest. Away from the area of interest the results of the model become less realistic with increasing distance to the area of interest.

The flow in the far-field does not have physical significance. For instance, the potential becomes unbounded far away if the total discharge in the near-field is not equal to zero. Physically it is not correct for an infinite aquifer that the potential becomes infinite. It is an artifact due to the fact that not the entire aquifer with all its boundaries and features is being modeled.

far-field

Next more will be said about the far-field. In the following the far-field is limited to the constant and the discharge at infinity; uniform flow and uniform extraction are excluded.

The far-field will be examined using a steady potential consisting of a constant and n wells. The wells are located at the points (x_k, y_k) and have a discharge Q_k. The constant is determined by the reference potential Φ_0 at the reference point (x_0, y_0). The potential at the point (x, y) is equal to

$$\overset{\text{steady}}{\Phi} = \Phi_0 + \sum_{k=1}^{n} \frac{Q_k}{4\pi} \ln(\frac{r_k^2}{R_k^2}) \tag{3.22}$$

where r_k is the distance from well k to the point (x, y)

$$r_k = \sqrt{(x - x_k)^2 + (y - y_k)^2} \tag{3.23}$$

and the length R_k in the potential for a steady well (3.4) is set equal to the distance between the well and the reference point

$$R_k = \sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2} \tag{3.24}$$

The constant in the potential for the far-field is only of interest if the sum of the discharges of the elements is equal to zero, otherwise the potential is unbounded. It is advantageous to define the potentials for the elements in such a way that the limit for $\sqrt{x^2 + y^2} \to \infty$ of the sum of the elements is equal to zero, if the sum of the discharges is equal to zero. In that case the constant is equal to the value of the potential at infinity. In a transient model the potential at the reference point may change, but with the transient elements that will be derived in chapter 4, the value at infinity does not change (it will be shown there that the potentials for those transient elements vanish in the limit for $\sqrt{x^2 + y^2} \to \infty$).

Potentials with the desired property are obtained, if the lengths R in the potentials for the wells are normalized with respect to a length L. The potential (3.22) can be written as

$$\begin{aligned} \overset{\text{steady}}{\Phi} &= \Phi_0 - \sum_{k=1}^{n} \frac{Q_k}{4\pi} \ln(\frac{R_k^2}{L^2}) + \sum_{k=1}^{n} \frac{Q_k}{4\pi} \ln(\frac{r_k^2}{L^2}) \\ &= \mathcal{C} + \sum_{k=1}^{n} \frac{Q_k}{4\pi} \ln(\frac{r_k^2}{L^2}) \end{aligned} \tag{3.25}$$

where the constant in the potential, $\mathcal{C}$, is equal to

$$\mathcal{C} = \Phi_0 - \sum_{k=1}^{n} \frac{Q_k}{4\pi} \ln(\frac{R_k^2}{L^2}) \tag{3.26}$$

This part of the potential is independent of x, y so that the discharge vector is fully described by the sum of the normalized potentials of the wells.

The far-field corresponding to the potential (3.25) is equal to, using $\sqrt{x^2+y^2} = r$

$$\begin{aligned} \overset{\text{steady}}{\Phi} &= \mathcal{C} + \frac{\sum_{k=1}^{n} Q_k}{4\pi} \ln(\frac{r^2}{L^2}) \\ &= \mathcal{C} + \frac{\mathcal{Q}}{4\pi} \ln(\frac{r^2}{L^2}) \qquad r \to \infty \end{aligned} \tag{3.27}$$

where $\mathcal{Q}$ is equal to the sum of the discharges in the near-field.

$$\mathcal{Q} = \sum_{k=1}^{n} Q_k \tag{3.28}$$

and the constant is indeed equal to the potential at infinity if the sum of the discharges is equal to zero.

In figure 3.7 equipotentials are given for a case with eight wells. The locations of the wells are indicated by arrows with the discharges. The origin for the polar coordinates r, θ is indicated by a cross.

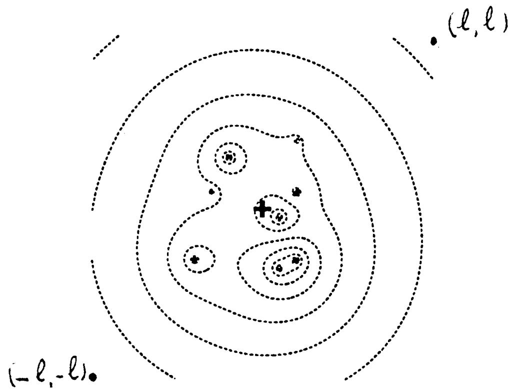

figure 3.7. *Equipotentials for a case with eight wells*

In figure 3.8 the same potential is plotted against the radial coordinate r for different values of the angular coordinate θ. The value of $\mathcal{C} + \frac{\mathcal{Q}}{4\pi} \ln(r^2/L^2)$ is marked in the graph by a dotted line. It can be seen that the curves for the potential for every θ converge to this line.

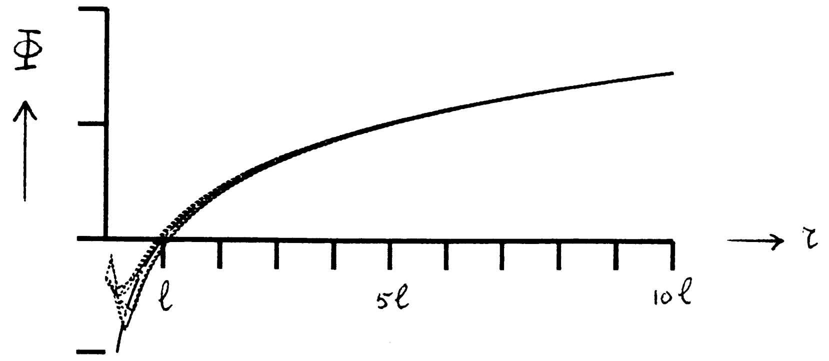

figure 3.8. *Potential as function of the radius r for several values of the angle θ*

If the length L is chosen to be equal to a value much larger than the distances of the wells to the origin the potential at $r = L$ can be apprioximated by (3.27), so that the potential at the circle of radius L is approximately equal to $\mathcal{C}$

$$\Phi\big|_{r=L} \simeq \mathcal{C} \tag{3.29}$$

and the radial discharge is close to (3.27)

$$\begin{aligned} Q_r\big|_{r=L} &\simeq -\frac{\partial}{\partial r}\{\mathcal{C} + \frac{Q}{4\pi}\ln(\frac{r^2}{L^2})\}\big|_{r=L} \\ &\simeq -\frac{Q}{2\pi L} \end{aligned} \tag{3.30}$$

This way the far-field could be visualized as a circular boundary of the problem where the potential and normal derivative are equal to (3.29) and (3.30) respectively (see figure 3.9).

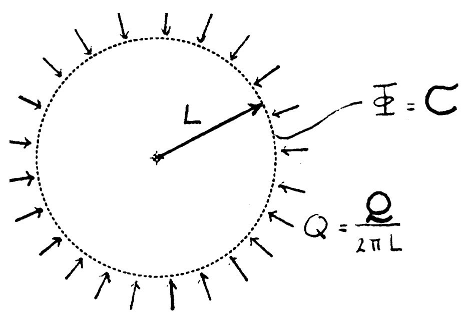

figure 3.9. *Visualization of far-field as imaginary boundary*

The values at the circle can be seen as the representation in the model of the influence of the aquifer features that are not included in the form of elements Those left out aquifer features control the level far away and provide recharge to the near-field.

This outer boundary is completely different from the boundary that is put on the outside of for instance a boundary element model, where an explicit boundary condition is specified on the outer boundary. In an analytic element model the values on the circle are not specified but follow implicitly from the boundary conditions that each represent an aquifer feature, plus the reference.

In the above the far-field has been discussed using a potential with no elements but wells. The discussion can be extended easily to other cases. The equivalent of (3.22) for a potential that contains the potentials for a constant, wells, line-sinks and area-sinks can be written as

$$\overset{\text{steady}}{\Phi} = \underset{c}{\Phi} + \sum_{k=1}^{n} \underset{k}{\overset{\text{steady}}{\Phi}} \tag{3.31}$$

where $\underset{k}{\overset{\text{steady}}{\Phi}}$ is the potential of element number k in which the value of the length R_k is chosen such that the potential $\underset{k}{\overset{\text{steady}}{\Phi}}$ is equal to zero at the reference point (x_0, y_0). If the potentials of all elements are normalized with respect to the length L, then the equivalent of (3.26) is

$$\overset{\text{steady}}{\Phi} = \mathcal{C} + \sum_{k=1}^{n} \{ \underset{k}{\overset{\text{steady}}{\Phi}} |_{R_k = L} \} \tag{3.32}$$

Since the behavior at infinity of elements discussed here is equal to that of a well with the same total discharge (see (3.11) for instance) the far-field of (3.32) is equal to

$$\begin{aligned}
\overset{\text{steady}}{\Phi} &= \mathcal{C} + \lim_{r \to \infty} \sum_{k=1}^{n} \{ \underset{k}{\overset{\text{steady}}{\Phi}} |_{R_k = L} \} \\
&= \mathcal{C} + \sum_{k=1}^{n} \{ \frac{\underset{k}{\overset{total}{Q}}}{4\pi} \ln(\frac{r^2}{L^2}) \} \\
&= \mathcal{C} + \frac{\mathcal{Q}}{4\pi} \ln(\frac{r^2}{L^2}) \qquad (r \to \infty)
\end{aligned} \tag{3.33}$$

which can be written as

$$\overset{\text{steady}}{\Phi} = \mathcal{C} + \frac{\mathcal{Q}}{4\pi} \ln(\frac{x^2 + y^2}{L^2}) \qquad (\sqrt{x^2 + y^2} \to \infty) \tag{3.34}$$

where $\mathcal{Q}$ is equal to the sum of the discharges

$$\mathcal{Q} = \sum_{k=1}^{n} \underset{k}{\overset{total}{Q}} \tag{3.35}$$

This is the same expression as was found for the far-field in the case with only wells. So that the above discussion for the far-field for that case can indeed be translated to more general cases.

4. transient analytic elements

Functions associated with transient analytic elements are derived in this chapter. These elements are the transient versions of the steady elements presented in chapter 3. Their behavior is similar and the same names will be used. The term order will be used in the same way also. The order of an elements refers to the variation in space of the strength of the element (Strack,1988). The term degree is introduced to indicate the way the strength changes as a function of time: a strength of degree zero is constant in time, degree one linear, degree two quadratic and so on.

The strengths will be equal to zero before the starting time t_0 of the elements. The potentials that will be derived for the elements apply to values of the time larger than the starting time $t > t_0$. The potential before the starting time is equal to zero

$$\Phi\Big|_{t\leq t_0} = 0 \tag{4.1}$$

which will be assumed in the following, and not mentioned with the expression for every potential.

The transient elements are solutions to either the heat equation (2.21), or the heat equation with a sink term (2.20). The domain is the entire plane minus an isolated point, or an isolated line-segment, or a region. These are the internal boundaries of the elements. The boundary conditions are specific for the particular element. The condition far away in space and the condition for large values of the time are more general. These will be referred to as the condition at infinity and the final condition respectively. The condition at infinity is that the influence of the element vanishes at infinity; the potential has to remain zero there

$$\lim_{\sqrt{x^2+y^2}\to\infty} \Phi = 0 \tag{4.2}$$

The starting time of the elements t_0, will be set equal to zero in the derivation of the transient elements. Therefor, the initial condition for all solutions is that the potential is equal to zero at time zero

$$\lim_{t\to 0} \Phi = 0 \tag{4.3}$$

A final condition is only applied to those solutions for which the boundary values remain bounded. The functions with a constant strength in time should converge to their steady counterparts for large values of the time. Two different final conditions will be used. These will be referred to as the strong and the weak final condition. The strong final condition requires that the transient potential approaches the potential of the corresponding steady element

$$\lim_{t\to\infty} \Phi = \overset{\text{steady}}{\Phi} \tag{4.4}$$

The strong condition is the condition that applies to any transient state for which the boundary conditions do not change any more after a certain time. The condition has to be applied to the sum of the potentials of the elements used to represent that transient state, but not necessarrily to the individual elements. For some elements the final condition is weakened to the condition that the discharges of the transient element converge to discharges of the corresponding steady element

$$\begin{cases} \lim_{t\to\infty} Q_x = \overset{\text{steady}}{Q_x} \\ \\ \lim_{t\to\infty} Q_y = \overset{\text{steady}}{Q_y} \end{cases} \tag{4.5}$$

If a function fulfills the strong final condition, it also fulfills the weak condition. However, if a function satisfies the weak final condition, the limit of the potential for large values of time is equal to the steady potential plus a function of the time. The function of time might even become infinite in the limit for $t \to \infty$.

In the following, the conditions are checked for each element, after its potential and discharges have been derived. It has been checked that the potentials fulfill the differential equation, but these checks are not included here. The fulfillment of the differential equation is checked by straight forward differentiation.

4.1 evapotranspiration functions

The potential for evapotranspiration is given in the category of functions that do not have singularities inside the plane. The potential fulfills the differential equation (2.20)

$$\frac{\partial^2}{\partial x^2}\Phi + \frac{\partial^2}{\partial y^2}\Phi = E + \frac{1}{\alpha}\frac{\partial}{\partial t}\Phi \tag{4.6}$$

A function, whose second spatial derivatives are equal to zero is chosen for the potential of evapotranspiration Φ_e

$$\frac{\partial^2}{\partial x^2}\Phi_e = \frac{\partial^2}{\partial y^2}\Phi_e = 0 \tag{4.7}$$

where the subscript $_e$ indicates evapotranspiration. The extraction of the potential (4.7) is equal to

$$E_e = -\frac{1}{\alpha}\frac{d}{dt}\Phi_e \tag{4.8}$$

Extraction that is constant in space is referred to as evapotranspiration of order one, which is consistent with Strack(1988). The order is two for an extraction that varies linearly in space. The degree of the strength indicates the way it changes in time.

An example of evapotranspiration of order 1 and degree n is

$$E_e = E_n t^n \tag{4.9}$$

so that the potential is given by

$$\Phi_e = -E_n \frac{\alpha}{n+1} t^{n+1} \tag{4.10}$$

This potential fulfills the differential equation and the initial condition. The potential is not equal to zero at infinity, so that the condition at infinity is not satisfied. The final condition only is applicable in case that the extraction remains bounded; i.e. for $n = 0$. The potential for $n = 0$ does not reach a limiting value for $t \to \infty$ so that the strong final condition (4.4) is not satisfied. The weak final condition (4.5) is not satisfied either since the discharges corresponding to (4.10) are identically equal to zero for all times while the discharges for steady evapotranspiration of order one are not equal to zero as can be seen from the fact that a pattern of equipotentials exists (see figure 3.1).

Evapotranspiration of order 2 corresponds to linear extraction

$$E_e = (E_{x,n} x + E_{y,n} y) t^n \tag{4.11}$$

and has a potential

$$\Phi_e = -(E_{x,n} x + E_{y,n} y) \frac{\alpha}{n+1} t^{n+1} \tag{4.12}$$

This potential also fulfills the differential equation (4.6) and the initial condition but not the condition at infinity. The final condition only is applicable in case that the extraction remains bounded; i.e. for $n = 0$. However, the final condition is not satisfied for $n = 0$.

4.1.1 application of evapotranspiration.

The potentials for evapotranspiration will be used to construct the potential for area-sinks (later in this chapter) and a far-field function (in chapter 5). They will not be used by themselves to model changing extraction out of the aquifer since they do not fulfill the conditions at infinity.

4.2 two elementary solutions

Carslaw and Jaeger (1986) give two elementary solutions of the heat equation (2.21) in an infinite domain: the instantaneous well (equation 10.3(1)) and the instantaneous doublet (equation 10.8(4)). The adjective instantaneous means that the strength is a Dirac δ function (see e.g. Wylie and Barrett, 1982, page 456) in time. The corresponding element with a continuous strength is obtained by integrating the instantaneous solution with respect to the time.

The potentials of the instantaneous well and doublet are indicated by $\underset{iw}{\Phi}$ and $\underset{id}{\Phi}$ and they have strengths $\underset{i}{Q}$ and $\underset{i}{s}$ respectively. The subscript i indicates an instantaneous quantity. The location has coordinates (x_0, y_0) and the instantaneous strength occurs at time t_0. The potentials $\underset{iw}{\Phi}$ and $\underset{id}{\Phi}$ are

$$\underset{iw}{\Phi} = -\frac{\underset{i}{Q}}{4\pi(t-t_0)} e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_0)}} \tag{4.13}$$

$$\underset{id}{\Phi} = \frac{\underset{i}{s}}{8\pi\alpha(t-t_0)^2}\{(x-x_0)\cos\theta + (y-y_0)\sin\theta\} e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_0)}} \tag{4.14}$$

where θ is the orientation of the doublet with respect to the positive x axis.

The following elements are derived from these two elementary solutions.

4.3 well

The potential for an instantaneous well is given on the previous page (4.13). Continuous wells are obtained by integrating an instantaneous well in time. The procedure is given by Carslaw and Jaeger(1986). The potential for wells with various strengths are available in the literature (see e.g. Carslaw and Jaeger, 1986, Jacob and Lohman, 1952, Abu-Zied and Scott, 1963, and Hantush, 1964).

4.3.1 well of degree zero.

The solution for a well of constant discharge after time zero, which is called a well of degree zero, is given by Carslaw and Jaeger (1986). It was first introduced into the field of groundwater flow by Theis (1935), who adapted it from the field of heat flow from a paper written by Carslaw.

The potential is equal to the sum of the potentials of instantaneous wells (4.13), whose discharges are equal to $\underset{0}{Q} d\tau$ and occur at time intervals $d\tau$. In the limit for $d\tau \to \infty$ the potential can be written as the following integral (Carslaw and Jaeger, 1986)

$$\underset{w0}{\Phi} = \int_0^t -\frac{\underset{0}{Q}}{4\pi(t-\tau)} e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-\tau)}} d\tau \tag{4.15}$$

so that

$$\underset{w0}{\Phi} = -\frac{\underset{0}{Q}}{4\pi} \mathrm{E}_1\left(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}\right) \tag{4.16}$$

where E_1 is the exponential integral which is defined as (Abramowitz and Stegun, 1972)

$$\mathrm{E}_1(x) = \int_x^\infty \frac{e^{-u}}{u} du \tag{4.17}$$

The discharges in x and y direction for the well with a constant strength are given by

$$\underset{w0}{Q}{}_x = -\frac{\underset{0}{Q}}{2\pi} \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}} \tag{4.18}$$

$$\underset{w0}{Q}{}_y = -\frac{\underset{0}{Q}}{2\pi} \frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}} \tag{4.19}$$

where (B.5) and (B.6) have been used. The radial discharge is found by means of (B.4)

$$\underset{w0}{Q}{}_r = -\frac{\underset{0}{Q}}{2\pi r} e^{-\frac{r^2}{4\alpha t}} \tag{4.20}$$

where r is the radial coordinate, centered at the location of the well (x_0, y_0).

4.3.2 checking well of degree zero.

Now that the potential (4.16) and the discharges (4.18), (4.19) and (4.20) of a well of degree zero have been derived, it is checked that the element indeed fulfills the applicable conditions (the check that the potential fulfills the differential equation is not included here, but is straightforward using the equations in appendix B).

4.3.2.1 initial condition. The exponential integral in equation (4.16) vanishes in the limit for the time approaching zero (B.14) so that the potential fulfills the initial condition of a zero value.

$$\lim_{t\to 0} \underset{w0}{\Phi} = \lim_{t\to 0}[-\frac{\underset{0}{Q}}{4\pi}\mathrm{E}_1(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t})] = 0 \tag{4.21}$$

4.3.2.2 boundary condition. The boundary condition for a well states that the total amount of water that is removed from the aquifer is equal to the discharge of the well. Using the expression for the radial discharge (4.20) it can be seen that the boundary condition (3.2) is satisfied

$$\begin{aligned}\lim_{r\to 0} -2\pi r \underset{w0}{Q_r} &= \lim_{r\to 0} -2\pi r[-\frac{\partial}{\partial r}\underset{w0}{\Phi}] \\ &= \lim_{r\to 0} -2\pi r[-\frac{\underset{0}{Q}}{2\pi}\frac{1}{r}e^{-\frac{r^2}{4\alpha t}}] \\ &= \underset{0}{Q}\end{aligned} \tag{4.22}$$

The condition at infinity (4.2) is also satisfied (see (B.13))

$$\lim_{\sqrt{x^2+y^2}\to\infty} \underset{w0}{\Phi} = 0 \tag{4.23}$$

And there is no flow from infinity (compare (3.8)):

$$\begin{aligned}\lim_{r\to\infty} 2\pi r \underset{w0}{Q_r} &= \lim_{r\to\infty} -\underset{0}{Q}e^{-\frac{r^2}{4\alpha t}} \\ &= 0\end{aligned} \tag{4.24}$$

where (4.20) is used.

4.3.2.3 final condition. The weak final condition (4.5) is satisfied. The discharge in x direction (4.18) approaches the steady x-component in (3.5) for $t \to \infty$

$$\begin{aligned}\lim_{t\to\infty} \underset{w0}{Q_x} &= \lim_{t\to\infty} -\frac{\underset{0}{Q}}{2\pi}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}} \\ &= -\frac{\underset{0}{Q}}{2\pi}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} \\ &= \overset{\text{steady}}{\underset{w}{Q_x}}\end{aligned} \tag{4.25}$$

and the discharge in y-direction (4.19) approaches the discharge in y-direction of the steady well (3.6)

$$\begin{aligned}\lim_{t\to\infty} \underset{w0}{Q_y} &= \lim_{t\to\infty} -\frac{\underset{0}{Q}}{2\pi}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}} \\ &= -\frac{\underset{0}{Q}}{2\pi}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} \\ &= \overset{\text{steady}}{\underset{w}{Q_y}}\end{aligned} \tag{4.26}$$

The potential (4.16) of the transient well does not approach the potential of a steady well (3.4), so that the strong final condition (4.4) is not satisfied. The limit for the time approaching infinity of the potential can be found using (B.15)

$$\begin{aligned}
\lim_{t\to\infty} \Phi_{w0} &= \lim_{t\to\infty} -\frac{Q_0}{4\pi}\mathrm{E}_1\Big(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}\Big) \\
&= \frac{Q_0}{4\pi}[\ln\Big(\frac{(x-x_0)^2+(y-y_0)^2}{R^2}\Big) - \ln\Big(\frac{4\alpha t}{R^2}\Big) + \gamma] \\
&= \Phi_w^{\text{steady}} + \frac{Q_0}{4\pi}[-\ln\Big(\frac{4\alpha t}{R^2}\Big) + \gamma]
\end{aligned} \tag{4.27}$$

where R is a length that was introduced in chapter 3. Thus the difference between the potential of a transient well and the potential of a steady well does not become zero for large values of time. In fact the potential for a constant transient well does not become steady. This is due to the fact that the value of the potential at infinity is kept at zero, which is a condition that makes a steady final state impossible.

4.3.3 application of well of degree zero.

The potential (4.16) is the potential for a well of degree zero, which start pumping at time zero. To obtain the potential for a well starting at time $t = t_0$, a translation in time is carried out

$$\Phi_{w0} = -\frac{Q_0}{4\pi}\mathrm{E}_1\Big(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_0)}\Big) \tag{4.28}$$

Now it is possible to make a well that is active for a limited time. The potential for a the well, which starts to pump a discharge Q at time t_1 and stops at time t_2, is equal to the potential for a well of discharge Q starting at t_1 plus the potential for a well starting at t_2 with a discharge $-Q$

$$\begin{aligned}
\Phi &= -\frac{Q_0}{4\pi}\mathrm{E}_1\Big(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_1)}\Big) - \frac{-Q_0}{4\pi}\mathrm{E}_1\Big(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_2)}\Big) \\
&= -\frac{Q_0}{4\pi}[\mathrm{E}_1\Big(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_1)}\Big) - \mathrm{E}_1\Big(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_2)}\Big)]
\end{aligned} \tag{4.29}$$

The discharge of the second well cancels the discharge of the first well, so that the apparent discharge is equal to zero after time t_2.

4.3.4 well of degree one.

The potential for a transient well of degree one is obtained by integrating the instantaneous well (4.13) with a strength that increases linearly in time. The potential for this well has not been found in the literature. The procedure is similar to that for the well of degree zero (4.16) that has been derived earlier in this chapter (see also Carslaw and Jaeger, 1986)

$$\begin{aligned}
\Phi_{w1} &= \int_0^t -\frac{Q_1\tau}{4\pi(t-\tau)} e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-\tau)}} d\tau \\
&= -\frac{Q_1 t}{4\pi}\{(1+\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t})\mathrm{E}_1\Big(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}\Big) - e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}}\} \\
&= -\frac{Q_1 t}{4\pi}\{\mathrm{E}_1\Big(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}\Big) - \mathrm{E}_2\Big(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}\Big)\}
\end{aligned} \tag{4.30}$$

where E_2 is defined as (Abramowitz and Stegun, 1972)

$$\begin{aligned} \mathrm{E}_2(x) &= e^{-x} - x\mathrm{E}_1(x) \\ \frac{d}{dx}\mathrm{E}_2(x) &= -\mathrm{E}_1(x) \end{aligned} \tag{4.31}$$

The discharge components for a well of degree one are equal to minus the components of the gradient of the potential

$$\underset{w1}{Q}_x = -\frac{\underset{1}{Q}t}{2\pi}\left\{\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}} - \frac{x-x_0}{4\alpha t}\mathrm{E}_1\left(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}\right)\right\} \tag{4.32}$$

$$\underset{w1}{Q}_y = -\frac{\underset{1}{Q}t}{2\pi}\left\{\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}} - \frac{y-y_0}{4\alpha t}\mathrm{E}_1\left(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha t}\right)\right\} \tag{4.33}$$

where (4.31), (B.5) and (B.6) have been used. Application of (B.4) gives the radial component of the discharge

$$\underset{w1}{Q}_r = -\frac{\underset{1}{Q}t}{2\pi}\left\{\frac{1}{r}e^{-\frac{r^2}{4\alpha t}} - \frac{r}{4\alpha t}\mathrm{E}_1\left(\frac{r^2}{4\alpha t}\right)\right\} \tag{4.34}$$

4.3.5 checking well of degree one.

Next the well of degree 1 is checked to determine whether the potential (4.30) and the discharges (4.32), (4.33) and (4.34) indeed fulfill the applicable conditions.

4.3.5.1 initial condition. The exponential integral goes to zero for very small values of the time (see equation (B.14). The exponential function in the potential (4.30) also vanishes. The initial condition that the potential is equal to zero at time zero thus is satisfied.

4.3.5.2 boundary condition. The boundary condition at the well (3.2) requires that the total radial discharge (4.34) into the element is equal to the discharge of the well $\underset{1}{Q}t$.

$$\begin{aligned} \lim_{r\to 0} -2\pi r\underset{w1}{Q}_r &= \lim_{r\to 0}\left[\underset{1}{Q}te^{-\frac{r^2}{4\alpha t}} - \underset{1}{Q}\frac{r^2}{4\alpha}\mathrm{E}_1\left(\frac{r^2}{4\alpha t}\right)\right] \\ &= \underset{1}{Q}t \end{aligned} \tag{4.35}$$

where equation (B.8) has been used to equate the term with the exponential integral to zero.

The boundary condition at infinity requires that the potential (4.30) goes to zero

$$\lim_{\sqrt{x^2+y^2}\to\infty} \underset{w1}{\Phi} = 0 \tag{4.36}$$

which is verified by (B.13) and (D.4).

4.3.6 application of well of degree one.

Wells of degree one can be used for the modeling of a well with a gradually changing discharge. The discharge then is continuous as opposed to the discontinuous discharge that would be the result of using wells of degree zero to model a transient well.

For instance a well can be made which has a discharge that grows linearly to its final value (see figure 4.1)

$$\begin{aligned} \Phi &= \underset{w1}{\Phi}_1 + \underset{w1}{\Phi}_2 \\ &= -\frac{\underset{1}{Q}(t-t_1)}{4\pi}\left\{\mathrm{E}_1\left(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_1)}\right) - \mathrm{E}_2\left(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_1)}\right)\right\} \\ &\quad - \frac{-\underset{1}{Q}(t-t_2)}{4\pi}\left\{\mathrm{E}_1\left(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_2)}\right) - \mathrm{E}_2\left(\frac{(x-x_0)^2+(y-y_0)^2}{4\alpha(t-t_2)}\right)\right\} \end{aligned} \tag{4.37}$$

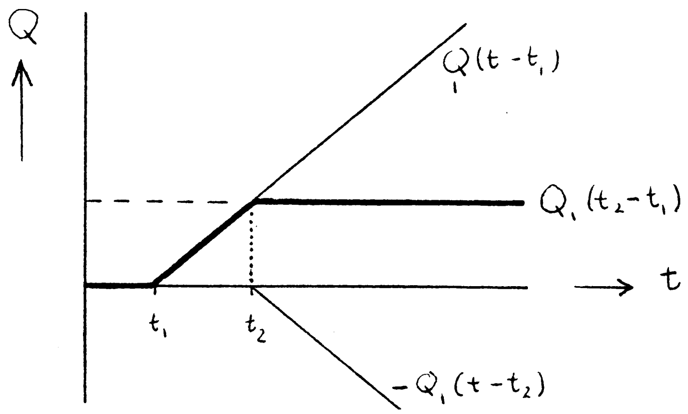

figure 4.1. *Application of wells of degree one*

4.4 line-sink

A line-sink is a sink with its discharge distributed along a line-segment. The distributed discharge σ is constant along a line-sink of order one. The discharge of a line-sink of degree n varies in time as a polynomial of order n in terms of $(t - t_0)$, where t_0 is the time the line-sink starts to discharge. The starting time is set equal to zero for the continuous elements in the following derivation. The procedure is the same as has been followed for the wells: the continuous elements are obtained from an instantaneous solution by means of integration.

The potentials for line-sinks are expressed in terms of the same local coordinates ξ, η as were used in chapter 3 (see figure 3.4). The end-points of the element are $(\overset{1}{\xi}, 0)$ and $(\overset{2}{\xi}, 0)$.

4.4.1 instantaneous line-sink.

The potential for an instantaneous line-sink is obtained from the potential of an instantaneous well, analog to the construction of a steady line-sink from a steady well (Strack, 1988). The line-segment from $(\overset{1}{\xi}, 0)$ to $(\overset{2}{\xi}, 0)$ is divided into intervals du with an instantaneous well in the center (see figure 4.2). The strength of the instantaneous wells is equal to $\underset{i}{\sigma} du$ and the discharge occurs at time t_0.

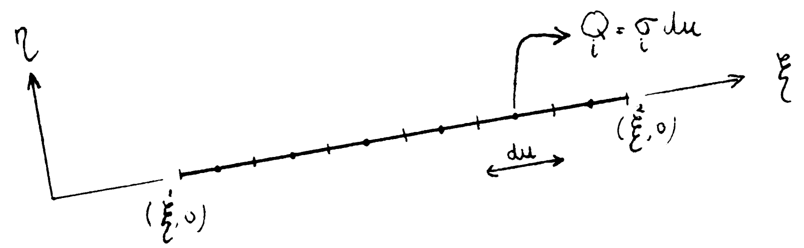

figure 4.2. *Derivation of instantaneous line-sink*

The potential of an instantaneous line-sink of order one is equal to the sum of the potentials of the instantaneous wells (4.13), all of the, same strength, in the limit for $du \to 0$. The resulting integral is

$$\begin{aligned}\underset{ils}{\Phi} &= -\frac{\underset{i}{\sigma}}{4\pi(t-t_0)} e^{-\frac{\eta^2}{4\alpha(t-t_0)}} \int_{\overset{1}{\xi}}^{\overset{2}{\xi}} e^{-\left(\frac{\xi-u}{2\sqrt{\alpha(t-t_0)}}\right)^2} du \\ &= -\underset{i}{\sigma} \frac{\sqrt{\alpha}}{4\sqrt{\pi}\sqrt{t-t_0}} e^{-\frac{\eta^2}{4\alpha(t-t_0)}} \{\mathrm{erfc}(\frac{1}{\sqrt{\alpha(t-t_0)}} \frac{\xi - \overset{2}{\xi}}{2}) - \mathrm{erfc}(\frac{1}{\sqrt{\alpha(t-t_0)}} \frac{\xi - \overset{1}{\xi}}{2})\}\end{aligned} \quad (4.38)$$

where t_0 is the time the instantaneous the line-sink discharges and erfc the complementary error-function, which is defined as (Abramowitz and Stegun, 1972)

$$\mathrm{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \quad (4.39)$$

Litkouhi and Beck give a function for an instantaneous semi-infinite line-source (Litkouhi and Beck (1982), equation (6)), which is closely related to (4.36). These authors use the function to derive a semi-infinite line-source of degree zero, which is the only function, that can be used for a line-sink in an infinite plane, that I have found in the literature.

Equation (4.38) will be used to derive the potential for a line-sink of arbitrary degree n. The potential for a line-sink of degree n will be given as a recursion formula. Subsequently, the potentials for line-sinks of degrees zero and one will be given explicitly.

4.4.2 line-sink of order zero and arbitrary degree.

A potential for a line-sink of degree n is equal to the sum of instantaneous line-sinks (4.38). Each instantaneous line-sink has a strength $\underset{i}{\sigma} = \underset{n}{\sigma}\tau^n d\tau$, where τ is the time at which the element discharges and $d\tau$ the time-interval in which one element discharges (see figure 4.3).

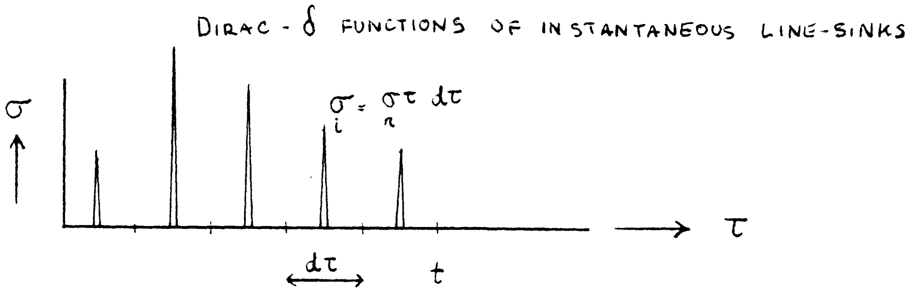

figure 4.3. *derivation of continuous line-sink*

If the limit is taken for $d\tau \to 0$ then the potential can be written as the following integral

$$\underset{lsn}{\Phi} = \int_0^t -\frac{\underset{n}{\sigma}\tau^n\sqrt{\alpha}}{4\sqrt{\pi}\sqrt{t-\tau}} e^{-\frac{\eta^2}{4\alpha(t-\tau)}} \{\mathrm{erfc}(\frac{\xi - \overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \mathrm{erfc}(\frac{\xi - \overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\}\, d\tau \tag{4.40}$$

wherein the lower bound of the integral is equal to zero, because the discharge of the continuous line-sink is starting at time zero.

The evaluation of the integral is given completely in appendix E. The integration involves a change of variables and repeated integration by parts. The change of variables is (E.4)

$$\begin{aligned} u &= \sqrt{\frac{\eta^2}{4\alpha(t-\tau)}} \\ t - \tau &= \frac{\eta^2}{4\alpha u^2} \\ d\tau &= \frac{\eta^2}{2\alpha u^3}\, du \end{aligned} \tag{4.41}$$

In order to keep track of the repeated integrations by parts the symbols $\overset{k}{I}_1$ and $\underset{j}{\overset{k}{I}_2}$ are introduced to denote two recurring integrals. The result, (E.20), is written in terms of these symbols also

$$\underset{lsn}{\Phi} = \frac{\underset{n}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}} \sum_{k=0}^{k=n} [\binom{n}{k}(-1)^{k+1} t^{n-k} (\frac{\eta^2}{4\alpha})^{k+\frac{1}{2}} \overset{k}{I}_1] \tag{4.42}$$

The term $\overset{k}{I}_1$ is given recursively. The term for $k = 0$ is equal to (E.17)

$$\begin{aligned} \overset{0}{I}_1 = & \sqrt{\frac{4\alpha t}{\eta^2}} e^{-\frac{\eta^2}{4\alpha t}} \{\mathrm{erfc}(\frac{\xi - \overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi - \overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\ & - \frac{1}{\sqrt{\pi}}\{\frac{\xi - \overset{2}{\xi}}{\eta}\mathrm{E}_1(\frac{(\xi - \overset{2}{\xi})^2 + \eta^2}{4\alpha t}) - \frac{\xi - \overset{1}{\xi}}{\eta}\mathrm{E}_1(\frac{(\xi - \overset{1}{\xi})^2 + \eta^2}{4\alpha t})\} \\ & - 2\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi - \overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi - \overset{1}{\xi}}{\eta})\}\, du \end{aligned} \tag{4.43}$$

the recursion formula is given by (E.10)

$$\begin{aligned}\overset{k}{I}_1 = {} & \frac{1}{2k+1}\left(\sqrt{\frac{4\alpha t}{\eta^2}}\right)^{2k+1} e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\ & + \frac{-1}{2k+1}\frac{2}{\sqrt{\pi}}\{\frac{\xi-\overset{2}{\xi}}{\eta}\underset{2}{\overset{k}{I}}_2 - \frac{\xi-\overset{1}{\xi}}{\eta}\underset{1}{\overset{k}{I}}_2\} \\ & + \frac{-2}{2k+1}\overset{k-1}{I}_1 \qquad\qquad k > 0\end{aligned} \tag{4.44}$$

where $\overset{k,j}{I}_2$ $(j = 1,2)$ is another recursively defined term. It is equal to (E.19) for $k = 0$

$$\underset{j}{\overset{0}{I}}_2 = \frac{1}{2}\mathrm{E}_1\left(\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}\right) \qquad j = 1,2 \tag{4.45}$$

The recursive relation is given by (E.14)

$$\begin{aligned}\underset{j}{\overset{k}{I}}_2 = {} & \frac{1}{2k}\left(\frac{4\alpha t}{\eta^2}\right)^k e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}} \\ & - \frac{1}{k}\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}\underset{j}{\overset{k-1}{I}}_2 \qquad j = 1,2 \qquad k > 0\end{aligned} \tag{4.46}$$

Two special cases of the general degree line-sink will be worked out after this solution has been checked: a line-sink with a constant strength after time zero (degree zero) and a line-sink with a linearly increasing strength (degree one).

4.4.3 checking line-sink of arbitrary degree.

The above potential derived for a line-sink of order one and degree n, was checked against the initial condition and the boundary condition. The details of the checking can be found in appendix E. The results are only summarized below. The final condition does not apply to elements which have a degree higher than zero. The check of the final condition for a line-sink with a degree zero is given after the special case of the line-sink of degree zero has been presented.

4.4.3.1 initial condition. The initial condition is that the potential (4.42) is equal to zero everywhere when the time is equal to zero. It is shown in appendix E that the initial condition is satisfied (see (E.29))

$$\lim_{t\to 0} \underset{lsn}{\Phi} = 0 \tag{4.47}$$

4.4.3.2 boundary condition. The boundary condition along a line-sink requires that the element removes an amount of water from the aquifer per unit length that is equal to the strength σ. The boundary condition for a line-sink of degree n with strength $\sigma = \underset{n}{\sigma}t^n$ can be stated as the condition that the component of the discharge normal to the element has a jump equal to the strength

$$\lim_{\eta\downarrow 0} Q_\eta - \lim_{\eta\uparrow 0} Q_\eta = -\underset{n}{\sigma}t^n \qquad \overset{1}{\xi} < \xi < \overset{2}{\xi} \tag{4.48}$$

which implies that the derivative with respect to η of the potential (4.42) is discontinuous across the element

$$\begin{cases} \lim_{\eta\downarrow 0} \frac{\partial}{\partial\eta}\underset{lsn}{\Phi} = \frac{1}{2}\underset{n}{\sigma}t^n \\ \lim_{\eta\uparrow 0} \frac{\partial}{\partial\eta}\underset{lsn}{\Phi} = -\frac{1}{2}\underset{n}{\sigma}t^n \end{cases} \qquad \overset{1}{\xi} < \xi < \overset{2}{\xi} \tag{4.49}$$

The derivative of the potential (4.42) with respect to η is equal to

$$\frac{\partial}{\partial\eta}\underset{lsn}{\Phi} = \frac{\underset{n}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}}\sum_{k=0}^{k=n}[\binom{n}{k}\frac{(-1)^{k+1}t^{n-k}}{(4\alpha)^{k+\frac{1}{2}}}\frac{\partial(\eta^{2k+1}\overset{k}{I}_1)}{\partial\eta}] \tag{4.50}$$

The terms for $k \neq 0$ do not contributes to this partial derivative for $\eta = 0$ (see (E.56)). The term for $k = 0$ gives (E.57)

$$\begin{cases} \lim_{\eta\downarrow 0}\frac{\partial}{\partial\eta}\underset{lsn}{\Phi} = \frac{1}{2}\underset{n}{\sigma}t^n \\ \lim_{\eta\uparrow 0}\frac{\partial}{\partial\eta}\underset{lsn}{\Phi} = -\frac{1}{2}\underset{n}{\sigma}t^n \end{cases} \tag{4.51}$$

which conforms to the boundary condition (4.49). Thus the potential for the line-sink of arbitrary degree n (4.42) fulfills the boundary condition at the element.

The condition at infinity requires that the potential is equal to zero at infinity

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty}\underset{lsn}{\Phi} = 0 \tag{4.52}$$

It is shown in appendix E that the condition at infinity is fulfilled (see (E.69) and (E.59)).

4.4.4 application of line-sinks of order one and degree n.

The general form of the strength for a line-sink of degree n is

$$\sigma = \underset{n}{\sigma}t^n + \underset{n-1}{\sigma}t^{n-1} + ... + \underset{1}{\sigma}t + \underset{0}{\sigma} \tag{4.53}$$

Thus the general potential for a line-sink of degree n is equal to the sum of $n+1$ line-sinks with degrees ranging from 0 to n

$$\underset{ls}{\overset{n}{\Phi}} = \sum_{k=0}^{k=n}\underset{lsk}{\Phi} \tag{4.54}$$

This line-sink can be used to approximate a straight boundary segment at which the potential is known at all times. Let $t = 0$ be the time that the transient flow starts, so that the transient discharge of the segment is equal to zero before time zero. To determine the $n+1$ coefficients of the strength of the line-sink, $\underset{k}{\sigma}$, $n+1$ times $\overset{s}{t}$ are selected. At each time the potential at the center of the line-sink is set equal to the known potential at that time Φ_k

$$\underset{ls}{\overset{n}{\Phi}}\Big|_{(\xi,\eta,t)=(0,0,\overset{s}{t})} = \Phi_k \qquad s = 1, 2, ...n, n+1 \tag{4.55}$$

which gives rise to a system of $n+1$ equations from which the $n+1$ unknowns $\underset{k}{\sigma}$ can be determined. Once the coefficients $\underset{k}{\sigma}$ have been determined the potential is known which approximates the specified potential at the boundary. It meets the specified values exactly at the center of the line-sink at the selected times $\overset{s}{t}$.

4.4.5 line-sink of order one and degree zero.

The potential for a line-sink with a constant strength $\underset{0}{\sigma}$ follows from (4.42) and (4.43) with the degree n equal to zero

$$\begin{aligned}\underset{ls0}{\Phi} = \underset{0}{\sigma}\Big[&-\frac{\sqrt{\alpha t}}{2\sqrt{\pi}} e^{-\frac{\eta^2}{4\alpha t}} \{\mathrm{erfc}(\frac{\xi - \overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi - \overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\ &+ \frac{\sqrt{\eta^2}}{4\pi}\{\frac{\xi - \overset{2}{\xi}}{\eta}\mathrm{E}_1(\frac{(\xi - \overset{2}{\xi})^2 + \eta^2}{4\alpha t}) - \frac{\xi - \overset{1}{\xi}}{\eta}\mathrm{E}_1(\frac{(\xi - \overset{1}{\xi})^2 + \eta^2}{4\alpha t})\} \\ &+ \frac{\sqrt{\eta^2}}{2\sqrt{\pi}} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2} \{\mathrm{erfc}(u\frac{\xi - \overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi - \overset{1}{\xi}}{\eta})\}\, du\Big]\end{aligned} \qquad (4.56)$$

A function closely related to this solution (4.56) has been derived by Litkouhi and Beck (1982) (equation (13)). They solved the problem of a semi-infinite body subjected to a constant heat flux over one half the surface, while the other half of the surface is insulated. The solution for the problem of a semi-infinite body subjected to a constant heat flux over an infinite strip can easily be obtained by superposition, as they have pointed out in the same paper. The solution can be imaged with respect to the surface to get the solution for an infinite body with an internal infinite strip source. The three dimensional solution can be reduced to a two dimensional one, since it is independent of the coordinate parallel to the strip. The resulting formula in two dimensions is a line-source in an infinite plane. The result is equivalent to the above potential for a line-sink (see figure 4.4).

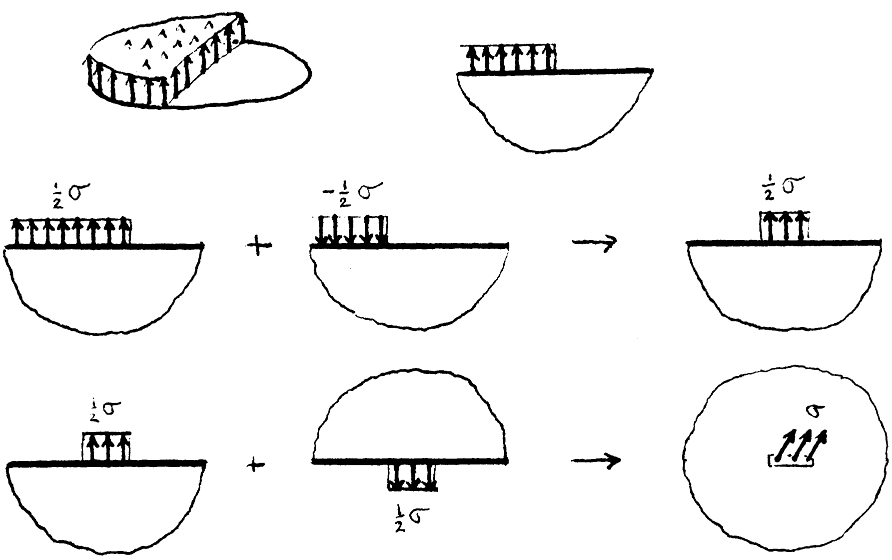

figure 4.4. *Solution of Litkouhi and Beck(1982)*

The discharge components of the line-sink with constant strength in directions parallel and normal to the element are found by taking the partial derivatives with respect to ξ and η. For the discharge component parallel to the line-sink the equations (A.5), (B.5), and (C.2) are used

$$\begin{aligned}\underset{ls0}{Q_\xi} &= -\frac{\partial}{\partial \xi}\underset{ls0}{\Phi} \\ &= -\frac{\underset{0}{\sigma}}{4\pi}\{\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t})-\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\}\end{aligned} \tag{4.57}$$

The expression for the discharge component normal to the line-sink is obtained using the equations (A.8), (B.6), and (C.3)

$$\begin{aligned}\underset{ls0}{Q_\eta} &= -\frac{\partial}{\partial \eta}\underset{ls0}{\Phi} \\ &= -\frac{\underset{0}{\sigma}}{2\sqrt{\pi}}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta})-\mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\,du\end{aligned} \tag{4.58}$$

4.4.6 checking solution for line-sink of degree zero.

It has been shown for the general case of a line-sink of arbitrary degree that the solution fulfills the initial conditions and the boundary conditions.

The solution for the line-sink with constant strength will be checked with another condition at infinity and the final condition. The latter was not checked for the line-sink of arbitrary degree since only applies to elements of degree zero. The former is the condition that the potential approaches the potential for a well of the same strength away from the line-sink. It was not checked for the line-sink of arbitrary degree since no wells of arbitrary degree were derived.

4.4.6.1 boundary condition. The condition at infinity for a line-sink (3.11) requires that far away from the line-sink the potential (4.56) approaches the potential of a well of degree zero (4.16) that removes the same amount of water from the aquifer. This is checked below using $\overset{m}{\xi}$ and $\Delta\xi$

as shorthand for $\frac{\overset{1}{\xi}+\overset{2}{\xi}}{2}$ and $(\overset{2}{\xi}-\overset{1}{\xi})$ respectively

$$
\begin{aligned}
\lim_{\sqrt{\xi^2+\eta^2}\to\infty}\underset{ls0}{\Phi} &= \lim_{\Delta\xi\to 0}\underset{ls0}{\Phi}\Big|_{\substack{\xi-\overset{2}{\xi}=\xi-\overset{m}{\xi}-\Delta\xi/2\\ \xi-\overset{1}{\xi}=\xi-\overset{m}{\xi}+\Delta\xi/2}}\\
&= \lim_{\Delta\xi\to 0}\underset{0}{\sigma}[-\frac{\sqrt{\alpha t}}{2\sqrt{\pi}}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})-\mathrm{erfc}(\frac{\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}\\
&\qquad+\frac{1}{4\pi}\{(\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2})\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2})^2+\eta^2}{4\alpha t})\\
&\qquad\qquad-(\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2})\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2})^2+\eta^2}{4\alpha t})\}\\
&\qquad+\frac{\eta}{2\sqrt{\pi}}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty}e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2}}{\eta})-\mathrm{erfc}(u\frac{\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2}}{\eta})\}\,du]\\
&=\frac{\partial}{\partial\xi}\Big[-\Delta\xi\underset{0}{\sigma}[-\frac{\sqrt{\alpha t}}{2\sqrt{\pi}}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{m}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}\\
&\qquad+\frac{1}{4\pi}\{(\xi-\overset{m}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t})\}\\
&\qquad+\frac{\eta}{2\sqrt{\pi}}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty}e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{m}{\xi}}{\eta})\}\,du]\Big]\\
&=\Delta\xi\underset{0}{\sigma}[\frac{\sqrt{\alpha t}}{2\sqrt{\pi}}e^{-\frac{\eta^2}{4\alpha t}}\frac{-1}{\sqrt{\pi\alpha t}}e^{-\frac{(\xi-\overset{m}{\xi})^2}{4\alpha t}}\\
&\qquad-\frac{1}{4\pi}\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t})\\
&\qquad-\frac{1}{4\pi}(-2)\frac{(\xi-\overset{m}{\xi})^2}{(\xi-\overset{m}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}}\\
&\qquad-\frac{\eta}{2\sqrt{\pi}}(-\frac{\eta}{\sqrt{\pi}})\frac{e^{-\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}}}{(\xi-\overset{m}{\xi})^2+\eta^2}]\\
&=\Delta\xi\underset{0}{\sigma}[-\frac{1}{4\pi}\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t})\\
&\qquad+e^{-\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}}\{-\frac{1}{2\pi}\}\\
&\qquad+\frac{e^{-\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}}}{(\xi-\overset{m}{\xi})^2+\eta^2}\{\frac{(\xi-\overset{m}{\xi})^2}{2\pi}+\frac{\eta^2}{2\pi}\}]\\
&=-\frac{\Delta\xi\underset{0}{\sigma}}{4\pi}\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t})\\
&=-\frac{\underset{ls0}{\overset{total}{Q}}}{4\pi}\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t})\\
&=\underset{w0}{\Phi}\Big|_{\underset{0}{Q}=\underset{ls0}{\overset{total}{Q}},\,x=\xi-\overset{m}{\xi},\,y=\eta}
\end{aligned}
\tag{4.59}
$$

where the definition of the derivative has been used

$$\frac{d}{dx} f = \lim_{\Delta x\to 0} \frac{f(x+\frac{\Delta x}{2}) - f(x-\frac{\Delta x}{2})}{\Delta x} \tag{4.60}$$

4.4.6.2 final condition. The strong final condition (4.4) that the potential of the line-sink of degree zero (4.56) has to approach the potential of a steady line-sink (3.13) is not satisfied. The limit for the time approaching infinity of the potential of the transient line-sink is derived below with equations (C.14), (B.15) and (A.24)

$$\begin{aligned}
\lim_{t\to\infty} \underset{ls0}{\Phi} = \lim_{t\to\infty} \underset{0}{\sigma}[& -\frac{\sqrt{\alpha t}}{2\sqrt{\pi}} e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
& + \frac{1}{4\pi}\{(\xi-\overset{2}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) - (\xi-\overset{1}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
& + \frac{\eta}{2\sqrt{\pi}}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\, du] \\
=\underset{0}{\sigma}[& -\frac{\sqrt{\alpha}}{2\sqrt{\pi}}\frac{\overset{2}{\xi}-\overset{1}{\xi}}{\sqrt{\pi\alpha}} \\
& - \frac{1}{4\pi}\{(\xi-\overset{2}{\xi})\ln(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{R^2}) - (\xi-\overset{1}{\xi})\ln(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{R^2}) \\
& \quad - (\xi-\overset{2}{\xi})(\ln(\frac{4\alpha t}{R^2})-\gamma) + (\xi-\overset{1}{\xi})(\ln(\frac{4\alpha t}{R^2})-\gamma)\} \\
& + \frac{\eta}{2\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\{\arctan(\frac{\eta}{\xi-\overset{2}{\xi}}) - \arctan(\frac{\eta}{\xi-\overset{1}{\xi}})\}] \\
=\underset{0}{\sigma}[& -\frac{\overset{2}{\xi}-\overset{1}{\xi}}{2\pi} \\
& - \frac{1}{4\pi}\{(\xi-\overset{2}{\xi})\ln(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{R^2}) - (\xi-\overset{1}{\xi})\ln\frac{((\xi-\overset{1}{\xi})^2+\eta^2}{R^2})\} \\
& - \frac{1}{4\pi}\{(\overset{2}{\xi}-\overset{1}{\xi})(\ln(\frac{4\alpha t}{R^2})-\gamma)\} \\
& + \frac{\eta}{2\pi}\{\arctan(\frac{\eta}{\xi-\overset{2}{\xi}}) - \arctan(\frac{\eta}{\xi-\overset{1}{\xi}})\}] \\
= \overset{\text{steady}}{\underset{ls}{\Phi}} & - \frac{\underset{0}{\sigma}}{4\pi}(\overset{2}{\xi}-\overset{1}{\xi})(\ln(\frac{4\alpha t}{R^2})-\gamma) \\
= \overset{\text{steady}}{\underset{ls}{\Phi}} & - \frac{\overset{total}{\underset{ls0}{Q}}}{4\pi}(\ln(\frac{4\alpha t}{R^2})-\gamma)
\end{aligned} \tag{4.61}$$

The weak final condition (4.5) is satisfied. The formula for the discharge component parallel to a transient line-sink (4.57) approaches the steady one (3.14) as can be seen by using (B.15)

$$\begin{aligned}
\lim_{t\to\infty} \underset{ls0}{Q_\xi} &= \lim_{t\to\infty} -\frac{\underset{0}{\sigma}}{4\pi}\{\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) - \mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
&= -\frac{\underset{0}{\sigma}}{4\pi}\{-\ln(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{R^2}) + \ln(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{R^2}) \\
&\qquad + \ln(\frac{4\alpha t}{R^2}) - \gamma - \ln(\frac{4\alpha t}{R^2}) + \gamma\} \\
&= \frac{\underset{0}{\sigma}}{4\pi}\{\ln(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{R^2}) - \ln(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{R^2})\} \\
&= \overset{\text{steady}}{\underset{ls}{Q_\xi}}
\end{aligned} \tag{4.62}$$

The limit for the discharge normal to the line-sink (4.58) is also equal to its steady counterpart (3.15) as can be seen from equation (A.24)

$$\begin{aligned}
\lim_{t\to\infty} \underset{ls0}{Q_\eta} &= -\frac{\underset{0}{\sigma}}{2\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\{\arctan(\frac{\eta}{\xi-\overset{2}{\xi}}) - \arctan(\frac{\eta}{\xi-\overset{1}{\xi}})\} \\
&= -\frac{\underset{0}{\sigma}}{2\pi}\{\arctan(\frac{\eta}{\xi-\overset{2}{\xi}}) - \arctan(\frac{\eta}{\xi-\overset{1}{\xi}})\} \\
&= \overset{\text{steady}}{\underset{ls}{Q_\eta}}
\end{aligned} \tag{4.63}$$

4.4.7 application of line-sinks of degree zero.

The expressions (4.56), (4.57) and (4.58) are given in terms of the local coordinates ξ and η. These have to be related to the global coordinates x and y in order to use these expressions for a line-sink at an arbitrary location. If the end-points of a line-sink are $(\overset{1}{x},\overset{1}{y})$ and $(\overset{2}{x},\overset{2}{y})$, the local coordiates can be set equal to

$$\xi = (x - \frac{\overset{1}{x}+\overset{2}{x}}{2})\cos\theta + (y - \frac{\overset{1}{y}+\overset{2}{y}}{2})\sin\theta \tag{4.64}$$

$$\eta = -(x - \frac{\overset{1}{x}+\overset{2}{x}}{2})\sin\theta + (y - \frac{\overset{1}{y}+\overset{2}{y}}{2})\cos\theta \tag{4.65}$$

where θ is equal to the orientation of the element

$$\theta = \arctan(\frac{\overset{2}{y}-\overset{1}{y}}{\overset{2}{x}-\overset{1}{x}}) \tag{4.66}$$

The transformation given by equations (4.64) and (4.65) maps the element on the ξ-axis with its center at the origin in the ξ,η plane, so that the η coordinates of the end-points are equal to zero, and the ξ coordinates

$$\overset{2}{\xi} = -\overset{1}{\xi} = \frac{1}{2}\sqrt{(\overset{2}{x}-\overset{1}{x})^2 + (\overset{2}{y}-\overset{1}{y})^2} \tag{4.67}$$

where $\sqrt{(\overset{2}{x}-\overset{1}{x})^2 + (\overset{2}{y}-\overset{1}{y})^2}$ is equal to the length of the element.

Line-sinks can be used to model creeks and other aquifer features that can be represented by line-segments along which the head is known. The strengths are a priori unknown. They are obtained using the condition that the head at center of the line-segment is equal to the specified value. The method of solving is covered in chapter 6.

4.4.8 line-sink of degree one.

The potential for a line-sink with linear strength $\sigma(t) = \underset{1}{\sigma} t$ starting at time zero is given by (4.42) with $n = 1$

$$\begin{aligned}
\underset{ls1}{\Phi} = \underset{1}{\sigma}[&\frac{1}{4\pi}(\xi - \overset{2}{\xi})\{\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{12\alpha} + \frac{\eta^2}{6\alpha} + t\}\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) \\
&- \frac{1}{4\pi}(\xi - \overset{1}{\xi})\{\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{12\alpha} + \frac{\eta^2}{6\alpha} + t\}\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}) \\
&+ \frac{\eta}{2\sqrt{\pi}}\{\frac{\eta^2}{6\alpha} + t\}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\,du \\
&- \frac{\sqrt{\alpha t}}{3\sqrt{\pi}}\{\frac{\eta^2}{4\alpha} + t\}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
&- \frac{t}{12\pi}\{(\xi-\overset{2}{\xi})e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}} - (\xi-\overset{1}{\xi})e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}\}]
\end{aligned} \tag{4.68}$$

The discharge in the direction parallel to a line-sink with a linear strength in time is equal to

$$\begin{aligned}
\underset{ls1}{Q_\xi} = \underset{1}{\sigma}[&- \frac{1}{4\pi}\{(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha} + t)\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) \\
&- (\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha} + t)\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
&+ \frac{t}{4\pi}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}\}]
\end{aligned} \tag{4.69}$$

where the equations (A.5), (B.5), (C.2), and (D.2) have been used. The discharge in the direction normal to the line-sink is given by

$$\begin{aligned}
\underset{ls1}{Q_\eta} = \underset{1}{\sigma}[&- \frac{\eta}{8\pi\alpha}\{(\xi-\overset{2}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) - (\xi-\overset{1}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
&- \frac{1}{2\sqrt{\pi}}\{\frac{\eta^2}{2\alpha} + t\}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\,du \\
&+ \frac{t\eta}{4\sqrt{\pi\alpha t}}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}]
\end{aligned} \tag{4.70}$$

where the equations (A.8), (B.6), (C.3), and (D.3) have been used.

4.4.9 checking line-sink of degree one.

It has been shown for the general case of a line-sink of arbitrary degree that the solution fulfills the differential equation, the initial conditions and the boundary conditions. An extra boundary condition is checked below

4.4.9.1 boundary condition. The condition at infinity for a line-sink (3.11) requires that far away from the line-sink the potential (4.68) should approach the potential of a well (4.30) that removes the same amount of water from the aquifer. This is checked below using (4.60) with the center of the line-sink indicated by $\overset{m}{\xi} = (\overset{2}{\xi} - \overset{1}{\xi})/2$ and the length by $\Delta\xi$

$$
\begin{aligned}
\lim_{\sqrt{\xi^2+\eta^2}\to\infty} \underset{ls1}{\Phi} &= \lim_{\Delta\xi\to 0} \underset{ls1}{\Phi}\Big|_{\substack{\xi-\overset{2}{\xi}=\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2}\\ \xi-\overset{1}{\xi}=\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2}}} \\
&= \lim_{\Delta\xi\to 0} \underset{1}{\sigma}[\frac{1}{4\pi}(\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2})\{\frac{(\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2})^2+\eta^2}{12\alpha}+\frac{\eta^2}{6\alpha}+t\}\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2})^2+\eta^2}{4\alpha t}) \\
&\quad -\frac{1}{4\pi}(\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2})\{\frac{(\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2})^2+\eta^2}{12\alpha}+\frac{\eta^2}{6\alpha}+t\}\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2})^2+\eta^2}{4\alpha t}) \\
&\quad +\frac{\eta}{2\sqrt{\pi}}\{\frac{\eta^2}{6\alpha}+t\}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2}}{\eta})-\mathrm{erfc}(u\frac{\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2}}{\eta})\}\,du \\
&\quad -\frac{\sqrt{\alpha t}}{3\sqrt{\pi}}\{\frac{\eta^2}{4\alpha}+t\}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})-\mathrm{erfc}(\frac{\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
&\quad -\frac{t}{12\pi}\{(\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2})e^{-\frac{(\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2})^2+\eta^2}{4\alpha t}}-(\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2})e^{-\frac{(\xi-\overset{m}{\xi}+\frac{\Delta\xi}{2})^2+\eta^2}{4\alpha t}}\}] \\
&= -\frac{\partial}{\partial\xi}\Big[\Delta\xi\underset{1}{\sigma}[\frac{1}{4\pi}(\xi-\overset{m}{\xi})\{\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{12\alpha}+\frac{\eta^2}{6\alpha}+t\}\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}) \\
&\quad +\frac{\eta}{2\sqrt{\pi}}\{\frac{\eta^2}{6\alpha}+t\}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{m}{\xi}}{\eta})\}\,du \\
&\quad -\frac{\sqrt{\alpha t}}{3\sqrt{\pi}}\{\frac{\eta^2}{4\alpha}+t\}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{m}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
&\quad -\frac{t}{12\pi}\{(\xi-\overset{m}{\xi})e^{-\frac{(\xi-\overset{m}{\xi}-\frac{\Delta\xi}{2})^2+\eta^2}{4\alpha t}}\}]\Big] \\
&= -\underset{ls1}{\overset{total}{Q}}[\frac{1}{4\pi}(\xi-\overset{m}{\xi})\{\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{12\alpha}+\frac{\eta^2}{6\alpha}+t\}\frac{-2(\xi-\overset{m}{\xi})}{(\xi-\overset{m}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}} \\
&\quad +\frac{1}{4\pi}\{\frac{(\xi-\overset{m}{\xi})^2}{4\alpha}+\frac{\eta^2}{12\alpha}+\frac{\eta^2}{6\alpha}+t\}\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}) \\
&\quad -\frac{\eta^2}{2\pi}\{\frac{\eta^2}{6\alpha}+t\}\frac{e^{-\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}}}{(\xi-\overset{m}{\xi})^2+\eta^2}+\frac{1}{3\pi}\{\frac{\eta^2}{4\alpha}+t\}e^{-\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}} \\
&\quad -\frac{t}{12\pi}e^{-\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}}+\frac{(\xi-\overset{m}{\xi})^2}{24\pi\alpha}e^{-\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}}] \\
&= -\frac{\underset{ls1}{\overset{total}{Q}}}{4\pi}[(1+\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t})\mathrm{E}_1(\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t})-e^{-\frac{(\xi-\overset{m}{\xi})^2+\eta^2}{4\alpha t}}] \\
&= \underset{w1}{\Phi}\Big|_{\underset{1}{Q}=\underset{ls1}{\overset{total}{Q}},\,x=\xi-\overset{m}{\xi},\,y=\eta}
\end{aligned}
\tag{4.71}
$$

4.5 line-doublet

To obtain the potential for a line-doublet, a point-doublet is integrated along a line-segment (Strack, 1988). The orientation of the point doublet is normal to the line-segment (see figure 4.5).

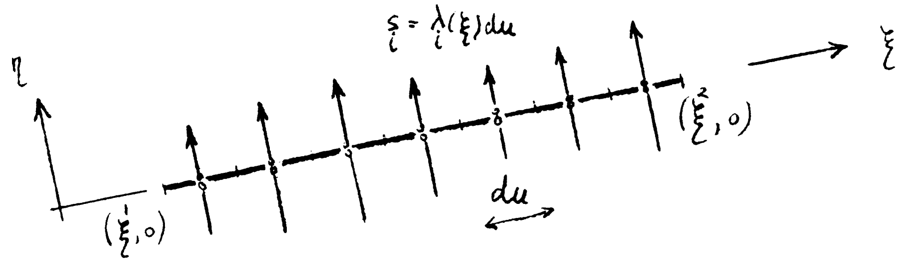

figure 4.5. *line-doublet.*

4.5.1 instantaneous line-doublet.

The potential for an instantaneous line-doublet of order zero can be calculated from instantaneous doublets (4.14) like the instantaneous line-sink (4.38) was derived from the instantaneous well (4.13). The instantaneous doublets are distributed along the line-segment from $(\overset{1}{\xi}, 0)$ to $(\overset{2}{\xi}, 0)$ at distances du and they have a strength $\underset{i}{s} = \underset{ic}{\lambda} du$ and orientation $\theta = \frac{\pi}{2}$. The the subscript db stands for line-doublet and i for an instantaneous quantity, while c indicates that the strength is constant with respect to the coordinate along the element ξ. The intregral resulting from the limit for $du \to 0$ is

$$\underset{icdb}{\Phi} = \int_{\overset{1}{\xi}}^{\overset{2}{\xi}} \frac{\underset{l0}{\lambda}}{8\pi\alpha(t-t_0)^2}\eta e^{-\frac{(\xi-u)^2+\eta^2}{4\alpha(t-t_0)}} du$$

$$= \frac{\underset{l0}{\lambda}}{8\pi\alpha(t-t_0)^2}\eta e^{-\frac{\eta^2}{4\alpha(t-t_0)}} \int_{\overset{1}{\xi}}^{\overset{2}{\xi}} e^{-\left(\frac{\xi-u}{2\sqrt{\alpha(t-t_0)}}\right)^2} du \qquad (4.72)$$

Recalling the definition of the error-function (4.39), the potential becomes

$$\underset{icdb}{\Phi} = \underset{ic}{\lambda} \frac{\eta}{8\sqrt{\pi\alpha}\,(t-t_0)^{\frac{3}{2}}} e^{-\frac{\eta^2}{4\alpha(t-t_0)}} \left\{\mathrm{erfc}\left(\frac{1}{\sqrt{\alpha(t-t_0)}}\frac{\xi-\overset{2}{\xi}}{2}\right) - \mathrm{erfc}\left(\frac{1}{\sqrt{\alpha(t-t_0)}}\frac{\xi-\overset{1}{\xi}}{2}\right)\right\} \qquad (4.73)$$

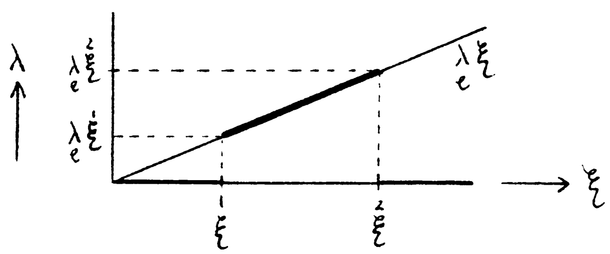

figure 4.6. *strength of linear doublet*

An instantaneous line-doublet of order one is derived from instantaneous doublets with a strength $s_i = \underset{il}{\lambda} u\, du$ where $(u, 0)$ is the location of the doublet, in the (ξ, η) coordinate system.

The subscript $\imath$ indicates that the strength is linear with respect to the coordinate along the element, ξ (see figure 4.6). The potential is equal to

$$
\begin{aligned}
\underset{ildb}{\Phi} &= \int_{\overset{1}{\xi}}^{\overset{2}{\xi}} \frac{\underset{il}{\lambda} u}{8\pi\alpha(t-t_0)^2}\eta e^{-\frac{(\xi-u)^2+\eta^2}{4\alpha(t-t_0)}}\,du \\
&= \frac{\underset{il}{\lambda}}{8\pi\alpha(t-t_0)^2}\eta e^{-\frac{\eta^2}{4\alpha(t-t_0)}} \int_{\overset{1}{\xi}}^{\overset{2}{\xi}} u e^{-(\frac{\xi-u}{2\sqrt{\alpha(t-t_0)}})^2}\,du \\
&= \frac{\underset{il}{\lambda}\eta e^{-\frac{\eta^2}{4\alpha(t-t_0)}}}{8\pi\alpha(t-t_0)^2} \int_{\overset{1}{\xi}}^{\overset{2}{\xi}} (\xi+(u-\xi)) e^{-(\frac{\xi-u}{2\sqrt{\alpha(t-t_0)}})^2}\,du \\
&= \frac{\underset{il}{\lambda}\eta e^{-\frac{\eta^2}{4\alpha(t-t_0)}}}{8\pi\alpha(t-t_0)^2}\Big[\xi \int_{\overset{1}{\xi}}^{\overset{2}{\xi}} e^{-(\frac{\xi-u}{2\sqrt{\alpha(t-t_0)}})^2}\,du - \int_{\overset{1}{\xi}}^{\overset{2}{\xi}} (\xi-u) e^{-(\frac{\xi-u}{2\sqrt{\alpha(t-t_0)}})^2}\,du\Big]
\end{aligned}
\tag{4.74}
$$

so that

$$
\begin{aligned}
\underset{ildb}{\Phi} &= \underset{il}{\lambda}\xi \frac{\eta}{8\sqrt{\pi}\sqrt{\alpha}(t-t_0)^{\frac{3}{2}}} e^{-\frac{\eta^2}{4\alpha(t-t_0)}}\{\mathrm{erfc}(\frac{1}{\sqrt{\alpha(t-t_0)}}\frac{\xi-\overset{2}{\xi}}{2}) - \mathrm{erfc}(\frac{1}{\sqrt{\alpha(t-t_0)}}\frac{\xi-\overset{1}{\xi}}{2})\} \\
&+ \underset{il}{\lambda}\frac{\eta}{4\pi(t-t_0)}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-t_0)}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-t_0)}}\}
\end{aligned}
\tag{4.75}
$$

Addition of (4.75) to (4.73) gives the following potential for the general form of an instantaneous line-doublet of order one

$$
\begin{aligned}
\underset{idb}{\Phi} &= (\underset{ic}{\lambda}+\xi\underset{il}{\lambda}) \frac{\eta}{8\sqrt{\pi}\sqrt{\alpha}(t-t_0)^{\frac{3}{2}}} e^{-\frac{\eta^2}{4\alpha(t-t_0)}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha(t-t_0)}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha(t-t_0)}})\} \\
&+ \underset{il}{\lambda}\frac{-\eta}{4\pi(t-t_0)}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-t_0)}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-t_0)}}\}
\end{aligned}
\tag{4.76}
$$

4.5.2 line-doublet of order one and degree zero.

The potential for a line-doublet of degree zero is obtained from the potentials of a series of instantaneous line-double same strength $(\underset{c0}{\lambda}+\xi\underset{l0}{\lambda})d\tau$, which becomes in the limit for $d\tau \to 0$

$$
\begin{aligned}
\underset{db0}{\Phi} &= \int_0^t \Big[(\underset{c0}{\lambda}+\xi\underset{l0}{\lambda}) \frac{\eta e^{-\frac{\eta^2}{4\alpha(t-\tau)}}}{8\sqrt{\pi}\sqrt{\alpha}(t-\tau)^{\frac{3}{2}}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\} \\
&+ \underset{l0}{\lambda}\frac{\eta}{4\pi(t-\tau)}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-\tau)}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-\tau)}}\}\Big]\,d\tau \\
&= (\underset{c0}{\lambda}+\xi\underset{l0}{\lambda}) \frac{\eta}{8\sqrt{\pi}\sqrt{\alpha}} \int_0^t \frac{e^{-\frac{\eta^2}{4\alpha(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\}d\tau \\
&+ \underset{l0}{\lambda}\frac{\eta}{4\pi}\int_0^t \frac{1}{t-\tau}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-\tau)}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-\tau)}}\}d\tau
\end{aligned}
\tag{4.77}
$$

The constant terms are taken outside the integral signs and the change of variables (4.41) is performed

$$\begin{aligned}
\underset{db0}{\Phi} &= (\underset{c0}{\lambda} + \xi\underset{l0}{\lambda})\frac{\eta}{8\sqrt{\pi}\sqrt{\alpha}}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty}(\frac{4\alpha u^2}{\eta^2})^{\frac{3}{2}}e^{-\frac{\eta^2}{4\alpha}\frac{4\alpha u^2}{\eta^2}} \\
&\qquad \{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha}}\sqrt{\frac{4\alpha u^2}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha}}\sqrt{\frac{4\alpha u^2}{\eta^2}})\}\frac{\eta^2}{2\alpha u^3}du \\
&\quad +\underset{l0}{\lambda}\frac{\eta}{4\pi}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty}\frac{4\alpha u^2}{\eta^2}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha}\frac{4\alpha u^2}{\eta^2}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha}\frac{4\alpha u^2}{\eta^2}}\}\frac{\eta^2}{2\alpha u^3}du \\
&= (\underset{c0}{\lambda} + \xi\underset{l0}{\lambda})\frac{\eta}{8\sqrt{\pi}\sqrt{\alpha}}\frac{4\sqrt{\alpha}}{\eta}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty}e^{-u^2}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}u) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}u)\}du \\
&\quad +\underset{l0}{\lambda}\frac{\eta}{4\pi}2\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty}\frac{1}{u}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{\eta^2}u^2} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{\eta^2}u^2}\}du
\end{aligned} \tag{4.78}$$

The first integral is the function (A.1), which is discussed in appendix A. The second integral can be written as

$$\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty}\frac{1}{u}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{\eta^2}u^2} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{\eta^2}u^2}\}du = \underset{k}{\overset{2}{I}}{}_2\big|_{k=0} - \underset{k}{\overset{1}{I}}{}_2\big|_{k=0} \tag{4.79}$$

where (E.11) has been used. Substituting the expression (4.45) for $\underset{j}{\overset{0}{I}}{}_2$ the potential for the line-doublet becomes

$$\begin{aligned}
\underset{db0}{\Phi} &= (\underset{c0}{\lambda} + \xi\underset{l0}{\lambda})\frac{1}{2\sqrt{\pi}}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty}e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\,du \\
&\quad +\underset{l0}{\lambda}[-\frac{\eta}{4\pi}\{\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) - \mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\}]
\end{aligned} \tag{4.80}$$

The discharges of a line-doublet of order one (linear in space) and degree zero (constant in time) are obtained by taking partial derivatives of the potential (4.80). The equations (A.5) and (B.5) are used to derive the ξ-component of the discharge vector

$$\begin{aligned}
\underset{db0}{Q}{}_\xi &= -\frac{\partial}{\partial\xi}\underset{db0}{\Phi} \\
&= (\underset{c0}{\lambda} + \xi\underset{l0}{\lambda})\frac{\eta}{2\pi}\{\frac{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}}}{(\xi-\overset{2}{\xi})^2+\eta^2} - \frac{e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}}{(\xi-\overset{1}{\xi})^2+\eta^2}\} \\
&\quad +\underset{l0}{\lambda}[-\frac{1}{2\sqrt{\pi}}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty}e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi})}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi})}{\eta})\}\,du \\
&\qquad -\frac{\eta}{2\pi}\{\frac{\xi-\overset{2}{\xi}}{(\xi-\overset{2}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}} \\
&\qquad\qquad -\frac{\xi-\overset{1}{\xi}}{(\xi-\overset{1}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}\}]
\end{aligned} \tag{4.81}$$

The discharge in η-direction is obtained using (A.8) and (B.6)

$$
\begin{aligned}
\underset{db0}{Q_\eta} &= -\frac{\partial}{\partial\eta}\underset{db0}{\Phi} \\
&= (\underset{c0}{\lambda}+\xi\underset{l0}{\lambda}[-\frac{1}{2\pi}\{\frac{\xi-\overset{2}{\xi}}{(\xi-\overset{2}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}} \\
&\qquad - \frac{\xi-\overset{1}{\xi}}{(\xi-\overset{1}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}\} \\
&\qquad + \frac{1}{4\sqrt{\pi}\sqrt{\alpha t}}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})-\mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}] \\
&\quad +\underset{l0}{\lambda}[\frac{1}{4\pi}\{\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t})-\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
&\qquad -\frac{\eta^2}{2\pi}\{\frac{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}}}{(\xi-\overset{2}{\xi})^2+\eta^2}-\frac{e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}}{(\xi-\overset{1}{\xi})^2+\eta^2}\}]
\end{aligned}
\tag{4.82}
$$

4.5.3 checking line-doublet of order one and degree zero.

The line-doublet of order one and degree zero, for which the potential (4.80) and the expressions for the discharge (4.81) and (4.82) have just been derived, will be checked now.

4.5.3.1 initial condition. It can be seen from (A.23) and (B.14) that the initial condition is fulfilled that the potential is equal to zero at time zero.

4.5.3.2 boundary condition. It follows from (A.17) that the boundary condition of a jump in potential across the element equal to the strength $\underset{c0}{\lambda}+\xi\underset{l0}{\lambda}$ is satisfied

$$
\begin{aligned}
\lim_{\eta\downarrow 0}\underset{db0}{\Phi}-\lim_{\eta\uparrow 0}\underset{db0}{\Phi} &= (\underset{c0}{\lambda}+\xi\underset{l0}{\lambda})\frac{1}{2\sqrt{\pi}}\sqrt{\pi}-(\underset{c0}{\lambda}+\xi\underset{l0}{\lambda})\frac{1}{2\sqrt{\pi}}(-\sqrt{\pi}) \\
&= \underset{c0}{\lambda}+\xi\underset{l0}{\lambda} \qquad\qquad \xi_1<\xi<\xi_2
\end{aligned}
\tag{4.83}
$$

The values of the potential (4.80) and the components (4.81) and (4.82) of the discharge vector are equal to zero at infinity as follows from the limits (A.22) and (B.13), so that the doublet does not have an influence at infinity and the condition at infinity is satisfied.

4.5.3.3 final condition. The strong final condition (4.4) is satisfied that the potential of a line-doublet of order one and degree zero approaches the discharge of a steady first order line-doublet. The limit for the time going to infinity of the potential (4.80) can be evaluated using (A.24) and

(B.15)

$$
\begin{aligned}
\lim_{t\to\infty} \underset{db0}{\Phi} &= (\underset{c0}{\lambda} + \xi \underset{l0}{\lambda})\left[\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\{\arctan(\frac{\eta}{\xi - \overset{2}{\xi}}) - \arctan(\frac{\eta}{\xi - \overset{1}{\xi}})\}\right] \\
&\quad + \underset{l0}{\lambda}\left[\frac{\eta}{4\pi}\{\ln(\frac{(\xi - \overset{2}{\xi})^2 + \eta^2}{R^2}) - \ln(\frac{(\xi - \overset{1}{\xi})^2 + \eta^2}{R^2}) \right. \\
&\qquad \left. - \ln(\frac{4\alpha t}{R^2}) + \gamma + \ln(\frac{4\alpha t}{R^2}) - \gamma\}\right] \\
&= (\underset{c0}{\lambda} + \xi \underset{l0}{\lambda})\left[\frac{1}{2\pi}\{\arctan(\frac{\eta}{\xi - \overset{2}{\xi}}) - \arctan(\frac{\eta}{\xi - \overset{1}{\xi}})\}\right] \\
&\quad + \underset{l0}{\lambda}\left[\frac{\eta}{4\pi}\{\ln(\frac{(\xi - \overset{2}{\xi})^2 + \eta^2}{R^2}) - \ln(\frac{(\xi - \overset{1}{\xi})^2 + \eta^2}{R^2})\}\right] \\
&= \overset{\text{steady}}{\underset{db}{\Phi}}
\end{aligned}
\tag{4.84}
$$

which is given in equation (3.19).

4.5.4 line-doublet of order and degree one.

The potential for a line-doublet with a strength, which varies linearly in time is obtained by the same procedure as the one of degree zero. Now the instantaneous line-doublets have different strengths, increasing linearly with the time τ at which the instantaneous doublet occurs. The strengths are equal to $(\underset{c1}{\lambda} + \xi \underset{l1}{\lambda})\tau \, d\tau$

$$
\begin{aligned}
\underset{db1}{\Phi} &= \int_0^t \left[(\underset{c1}{\lambda} + \xi \underset{l1}{\lambda})\tau \frac{\eta e^{-\frac{\eta^2}{4\alpha(t-\tau)}}}{8\sqrt{\pi}\sqrt{\alpha}(t-\tau)^{\frac{3}{2}}}\{\operatorname{erfc}(\frac{\xi - \overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \operatorname{erfc}(\frac{\xi - \overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\}\right. \\
&\quad \left. + \underset{l1}{\lambda}\tau \frac{\eta}{4\pi(t-\tau)}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-\tau)}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-\tau)}}\}\right] d\tau \\
&= \frac{\underset{c1}{\lambda} + \xi \underset{l1}{\lambda}}{8\sqrt{\pi\alpha}}\eta \int_0^t \frac{t + (\tau - t)}{(t-\tau)^{\frac{3}{2}}} e^{-\frac{\eta^2}{4\alpha(t-\tau)}}\{\operatorname{erfc}(\frac{\xi - \overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \operatorname{erfc}(\frac{\xi - \overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\} d\tau \\
&\quad + \underset{l1}{\lambda}\frac{\eta}{4\pi}\int_0^t \frac{t + (\tau - t)}{t - \tau}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-\tau)}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-\tau)}}\} d\tau \\
&= \frac{\underset{c1}{\lambda} + \xi \underset{l1}{\lambda}}{8\sqrt{\pi\alpha}}\eta t \int_0^t \frac{e^{-\frac{\eta^2}{4\alpha(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}\{\operatorname{erfc}(\frac{\xi - \overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \operatorname{erfc}(\frac{\xi - \overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\} d\tau \\
&\quad + \underset{l1}{\lambda}\frac{\eta t}{4\pi}\int_0^t \frac{1}{t - \tau}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-\tau)}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-\tau)}}\} d\tau \\
&\quad - (\underset{c1}{\lambda} + \xi \underset{l1}{\lambda})\frac{\eta}{8\sqrt{\pi\alpha}}\int_0^t \frac{e^{-\frac{\eta^2}{4\alpha(t-\tau)}}}{\sqrt{t-\tau}}\{\operatorname{erfc}(\frac{\xi - \overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \operatorname{erfc}(\frac{\xi - \overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\} d\tau \\
&\quad - \underset{l1}{\lambda}\frac{\eta}{4\pi}\int_0^t \{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-\tau)}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-\tau)}}\} d\tau
\end{aligned}
\tag{4.85}
$$

The first two integrals in this are the same as the integrals in (4.77) in the derivation of the line-doublet of degree zero. The third and fourth integral are evaluated using again the change of variables (4.41). The third integral becomes

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{t-\tau}} e^{-\frac{\eta^2}{4\alpha(t-\tau)}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\}d\tau \\
&\qquad = \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} \frac{2\sqrt{\alpha}u}{\eta} e^{-u^2}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}u) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}u)\}\frac{\eta^2}{2\alpha u^3}du \\
&\qquad = \frac{\eta}{\sqrt{\alpha}}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} \frac{1}{u^2} e^{-u^2}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}u) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}u)\}du \\
&\qquad = \frac{\eta}{\sqrt{\alpha}}\overset{k}{I}_1\big|_{k=0}
\end{aligned} \tag{4.86}$$

where (E.6) has been used. Equation (4.43) is an expression for $\overset{0}{I}_1$, which can be substituted into (4.86).

The change of variables (4.41) changes the last integral of (4.85) into

$$\begin{aligned}
&\int_0^t \{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-\tau)}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-\tau)}}\}d\tau \\
&\qquad = \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} \{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{\eta^2}u^2} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{\eta^2}u^2}\}\frac{\eta^2}{2\alpha u^3}du \\
&\qquad = \frac{\eta^2}{2\alpha}\{\underset{2}{\overset{k}{I}}_2\big|_{k=1} - \underset{1}{\overset{k}{I}}_2\big|_{k=1}\}
\end{aligned} \tag{4.87}$$

where (E.11) has been used. Using (4.46) and (4.45), $\underset{j}{\overset{1}{I}}_2$ can be written as

$$\underset{j}{\overset{1}{I}}_2 = \frac{2\alpha t}{\eta^2}e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}} - \frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}\frac{1}{2}\mathrm{E}_1(\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}) \quad j=1,2 \tag{4.88}$$

Thus the potential for the line-doublet of order and degree one (4.85) can be written as

$$\begin{aligned}
\underset{db1}{\Phi} = (\underset{c1}{\lambda}+\xi\underset{l1}{\lambda})[&+\frac{\eta}{8\pi\alpha}\{(\xi-\overset{2}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) - (\xi-\overset{1}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
&+\frac{1}{2\sqrt{\pi}}\{\frac{\eta^2}{2\alpha}+t\}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi})}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi})}{\eta})\}\,du \\
&-\frac{\sqrt{t}\eta}{4\sqrt{\pi}\sqrt{\alpha}}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}] \\
+\underset{l1}{\lambda}[&-\frac{\eta}{4\pi}\{\{\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha}+t\}\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) \\
&\qquad -\{\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha}+t\}\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
&+\frac{t\eta}{4\pi}\{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}} - e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}\}]
\end{aligned} \tag{4.89}$$

The component in ξ direction of the discharge of the line-doublet of order and degree one is equal to

$$
\begin{aligned}
\underset{db1}{Q_\xi} &= -\frac{\partial}{\partial \xi}\underset{db1}{\Phi} \\
&= (\underset{c1}{\lambda} + \overset{2}{\xi}\underset{l1}{\lambda})\frac{\eta t}{2\pi}\frac{e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}}}{(\xi-\overset{2}{\xi})^2+\eta^2} \\
&-(\underset{c1}{\lambda} + \overset{1}{\xi}\underset{l1}{\lambda})\frac{\eta t}{2\pi}\frac{e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}}{(\xi-\overset{1}{\xi})^2+\eta^2} \\
&-(\underset{c1}{\lambda} + \xi\underset{l1}{\lambda})\frac{\eta}{8\pi\alpha}\{\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) - \mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
&\quad +\underset{l1}{\lambda}[-\frac{1}{2\sqrt{\pi}}\{\frac{\eta^2}{2\alpha}+t\}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi})}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi})}{\eta})\}\,du \\
&\qquad + \frac{\sqrt{t}\eta}{4\sqrt{\pi}\sqrt{\alpha}}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}] \qquad (4.90)
\end{aligned}
$$

where the equations (A.5), (B.5), (C.2), and (D.2) are used. The component in η direction of the discharge of the line-doublet of order and degree one is equal to

$$
\begin{aligned}
\underset{db1}{Q_\eta} &= -\frac{\partial}{\partial\eta}\underset{db1}{\Phi} \\
&= (\underset{c1}{\lambda} + \xi\underset{l1}{\lambda})[-\frac{t}{2\pi}\{\frac{\xi-\overset{2}{\xi}}{(\xi-\overset{2}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}} \\
&\qquad\qquad - \frac{\xi-\overset{1}{\xi}}{(\xi-\overset{1}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}\} \\
&\qquad - \frac{1}{8\pi\alpha}\{(\xi-\overset{2}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) \\
&\qquad\qquad - (\xi-\overset{1}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
&\qquad + \frac{\sqrt{t}}{2\sqrt{\pi}\sqrt{\alpha}}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
&\qquad - \frac{\eta}{2\sqrt{\pi}\alpha}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi})}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi})}{\eta})\}\,du] \\
&\quad + \underset{l1}{\lambda}[-\frac{t}{4\pi}\{(\frac{2\eta^2}{(\xi-\overset{2}{\xi})^2+\eta^2}+1)e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}} \\
&\qquad\qquad - (\frac{2\eta^2}{(\xi-\overset{1}{\xi})^2+\eta^2}+1)e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}\} \\
&\qquad + \frac{1}{4\pi}\{(\frac{1}{4\alpha}(\xi-\overset{2}{\xi})^2 + \frac{3}{4\alpha}\eta^2 + t)\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) \\
&\qquad\qquad - (\frac{1}{4\alpha}(\xi-\overset{1}{\xi})^2 + \frac{3}{4\alpha}\eta^2 + t)\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\}]
\end{aligned}
\tag{4.91}
$$

where the equations (A.8), (B.6), (C.3), and (D.3) are used.

4.5.5 checking line-doublet of order one and degree one.

The line-doublet of order one and degree one, (4.89), (4.90), and (4.91) will be checked.

4.5.5.1 initial condition. The potential (4.89) is zero at time zero, as follows from the limits (A.23), (B.14), (C.10) and (D.5).

4.5.5.2 boundary condition. The boundary condition that the jump in the potential at the line-doublet is equal to the strength is fulfilled as can be seen from the coefficient of the integral in (4.89).

The value of the potential (4.89) is equal to zero at infinity as follows from the limits (A.22) and (B.13), so that the doublet does not have an influence at infinity and the condition at infinity is satisfied.

4.5.6 application of line-doublets.

Line-doublets can be used to create to area-sinks, which will be explained later in this chapter.

4.6 area-sink

The potential for a transient area-sink is obtained in the same way as the way applied by Strack(1988) to obtain a general expression for the potential for a steady state area-sink. In this approach, the potential is split into two parts (Strack, 1988). A function is chosen for the first part that gives the extraction of the area-sink and no extraction outside the element. It fulfills the differential equation with extraction (2.20). The second part is then constructed such that it removes all the discontinuities that the first function may have. It satisfies the differential equation without extraction (2.21).

The boundary of the area-sink is taken as a polygon. The potential is written as

$$\underset{as}{\Phi} = \overset{1}{\Phi} + \overset{2}{\Phi} \tag{4.92}$$

where the index $_{as}$ stands for area-sink. The potential $\overset{1}{\Phi}$ is chosen to be equal to zero outside the area-sink and equal to a potential that gives extraction of the required degree and order inside

$$\overset{1}{\Phi} = \begin{cases} 0 & \text{outside} \\ \underset{E}{\Phi} & \text{inside} \end{cases} \tag{4.93}$$

which gives the extraction

$$\overset{1}{E} = \underset{as}{E} = \begin{cases} 0 & \text{outside} \\ \epsilon & \text{inside} \end{cases} \tag{4.94}$$

where ϵ is the strength of the area-sink. The potential $\overset{1}{\Phi}$ and the normal component of its gradient are discontinuous across the boundary of the area-sink, which is a polygon.

The potential $\overset{2}{\Phi}$ is the sum of the potentials for line-doublets and line-sinks along the sides of the polygon. The strengths of the line-doublets are chosen such that the jump in the potential $\overset{1}{\Phi}$ is canceled. The strengths the line-sinks are determined such that the discontinuity in the normal component of the gradient of $\overset{1}{\Phi}$ is eleminated. Thus the line-doublets are used to create a continuous potential and the line-sinks to let the normal discharge be continuous across the boundary of the area sink. The potential $\overset{2}{\Phi}$ may thus be written as follows for an area-sink with a boundary polygon that consists of n sides

$$\overset{2}{\Phi} = \sum_{j=1}^{n} \underset{db}{\Phi_j}\Big|_{\lambda=-\Delta\overset{1}{\Phi}} + \underset{ls}{\Phi_j}\Big|_{\sigma=-\Delta\overset{1}{Q}_n} \tag{4.95}$$

where $\overset{1}{Q}_n$ is the normal component of the discharge associated with the potential $\overset{1}{\Phi}$. It is taken positive in outward normal direction.

4.6.1 area-sink of order one and degree zero.

Strack (personal communication) suggested to use a function that is linear in time for the first part of the potential for an area-sink of order one and degree zero. The function is chosen as

$$\overset{1}{\Phi} = \begin{cases} 0 & \text{outside} \\ -\underset{0}{\epsilon}\alpha t & \text{inside} \end{cases} \tag{4.96}$$

This function is equal to the potential for uniform evaporation of degree zero (4.10) inside the area-sink, where the extraction is equal to $\underset{0}{\epsilon}$, the strength of the area-sink. And the the extraction due to (4.96) is equal to zero outside the area-sink, as it should be (see equation (4.94)). The jump across the boundary is constant in space and increases linearly in time. The jump can be eliminated by line-doublets, placed at the sides of the boundary polygon. The strengths of the line-doublets has to be constant along the element and linearly increasing in time, so that the potential $\overset{2}{\Phi}$ (4.95) is equal to the potential (4.89), for a line-doublet of order zero and degree one, along each side of the polygon

$$\begin{aligned}\overset{2}{\Phi} = \underset{0}{\epsilon}\alpha \sum_{j=1}^{n} \Big[& \frac{\eta_j}{8\pi\alpha}\{(\xi_j - \overset{2}{\xi}_j)\mathrm{E}_1(\frac{(\xi_j - \overset{2}{\xi}_j)^2 + \eta_j^2}{4\alpha t}) - (\xi_j - \overset{1}{\xi}_j)\mathrm{E}_1(\frac{(\xi_j - \overset{1}{\xi}_j)^2 + \eta_j^2}{4\alpha t})\} \\ & + \frac{1}{2\sqrt{\pi}}\{\frac{\eta_j^2}{2\alpha} + t\} \int_{\frac{\eta_j^2}{2\sqrt{\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi_j - \overset{2}{\xi}_j}{\eta_j}) - \mathrm{erfc}(u\frac{\xi_j - \overset{1}{\xi}_j}{\eta_j})\}\, du \\ & - \frac{\sqrt{t}\eta_j}{4\sqrt{\pi}\sqrt{\alpha}} e^{-\frac{\eta_j^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi_j - \overset{2}{\xi}_j}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi_j - \overset{1}{\xi}_j}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}\Big]\end{aligned} \tag{4.97}$$

where ξ_j and η_j are the local coordinates parallel and normal to side j. They are chosen such that η_j is positive inside the area-sink. The nodes at the ends of side j have local coordinates $(\overset{1}{\xi}_j, 0)$ and $(\overset{2}{\xi}_j, 0)$ as is shown in figure 4.7.

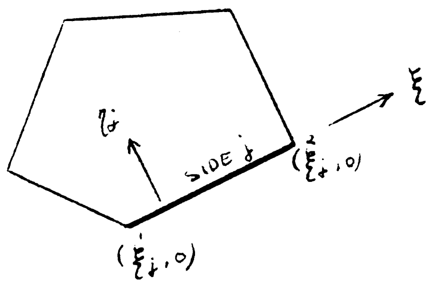

figure 4.7. *local coordinates at side j of an area-sink*

The complete potential of an area-sink with constant strength bounded by a polygon of n sides then becomes

$$\underset{as0}{\Phi} = \begin{cases} \overset{2}{\Phi} & \text{outside} \\ -\underset{0}{\epsilon}\alpha t + \overset{2}{\Phi} & \text{inside} \end{cases} \tag{4.98}$$

where the potential $\overset{2}{\Phi}$ is given by (4.97). components of the discharge vector are obtained by taking the The partial derivatives with respect to x and y of the potential $\overset{1}{\Phi}$ are equal to zero, so that it does not contribute to the discharge vector. Thus components of the discharge for the area-sink of order one are equal to the components of the discharge for the line-doublets that make up $\overset{2}{\Phi}$

$$\underset{as0}{Q}_x = \sum_{j=1}^{n} [\overset{j}{Q}_{\xi_j} \cos\theta_j - \overset{j}{Q}_{\eta_j} \sin\theta_j] \tag{4.99}$$

$$\underset{as0}{Q}_y = \sum_{j=1}^{n} [\overset{j}{Q}_{\xi_j} \sin\theta_j + \overset{j}{Q}_{\eta_j} \cos\theta_j] \tag{4.100}$$

where the components of the discharge for side j are determined from (4.90) and (4.91)

$$\overset{j}{Q}_{\xi_j} = \underset{0}{\epsilon}\alpha[\frac{\eta_j t}{2\pi}\{\frac{e^{-\frac{(\xi_j-\overset{2}{\xi}_j)^2+\eta_j^2}{4\alpha t}}}{(\xi_j-\overset{2}{\xi}_j)^2+\eta_j^2} - \frac{e^{-\frac{(\xi_j-\overset{1}{\xi}_j)^2+\eta_j^2}{4\alpha t}}}{(\xi_j-\overset{1}{\xi}_j)^2+\eta_j^2}\}$$
$$-\frac{\eta_j}{8\pi\alpha}\{\mathrm{E}_1(\frac{(\xi_j-\overset{2}{\xi}_j)^2+\eta_j^2}{4\alpha t}) - \mathrm{E}_1(\frac{(\xi_j-\overset{1}{\xi}_j)^2+\eta_j^2}{4\alpha t})\}] \qquad (4.101)$$

$$\overset{j}{Q}_{\eta_j} = \underset{0}{\epsilon}\alpha[-\frac{t}{2\pi}\{\frac{\xi_j-\overset{2}{\xi}_j}{(\xi_j-\overset{2}{\xi}_j)^2+\eta_j^2}e^{-\frac{(\xi_j-\overset{2}{\xi}_j)^2+\eta_j^2}{4\alpha t}}$$
$$-\frac{\xi_j-\overset{1}{\xi}_j}{(\xi_j-\overset{1}{\xi}_j)^2+\eta_j^2}e^{-\frac{(\xi_j-\overset{1_j}{\xi})^2+\eta_j^2}{4\alpha t}}\}$$
$$-\frac{1}{8\pi\alpha}\{(\xi_j-\overset{2}{\xi}_j)\mathrm{E}_1(\frac{(\xi_j-\overset{2}{\xi}_j)^2+\eta_j^2}{4\alpha t})$$
$$-(\xi_j-\overset{1}{\xi}_j)\mathrm{E}_1(\frac{(\xi_j-\overset{1}{\xi}_j)^2+\eta_j^2}{4\alpha t})\}$$
$$+\frac{\sqrt{t}}{2\sqrt{\pi}\sqrt{\alpha}}e^{-\frac{\eta_j^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi_j-\overset{2}{\xi}_j}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi_j-\overset{1}{\xi}_j}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}$$
$$-\frac{\eta_j}{2\sqrt{\pi}\alpha}\int_{\frac{\eta_j}{2\sqrt{\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi_j-\overset{2}{\xi}_j)}{\eta_j}) - \mathrm{erfc}(u\frac{\xi_j-\overset{1}{\xi}_j)}{\eta_j})\}\,du] \qquad (4.102)$$

4.6.2 checking area-sink of order one and degree zero.

4.6.2.1 continuity. The function for the area-sink was constructed in such a way that both the potential and the discharge are continuous across the boundary of the area-sink so that the change from one differential equation to an other does not cause discontinuities in the potential (and the head) and the discharge vector.

4.6.2.2 initial condition. The potential is zero at time zero for all parts, so that the sum is equal to zero, too.

4.6.2.3 boundary condition. The element has been constructed in such a way that the potential and the normal discharge are continuous. All parts of the potential fulfill the condition at infinity (4.2), so that the sum also is equal to zero at infinity.

4.6.2.4 final condition. It has only been verified numerically that the components of the discharge vector for a transient area-sink with four sides approach the components of the discharge vector for a steady area-sink of order one. Moreover it was verified numerically that for the potentials the following relation is valid

$$\lim_{t\to\infty} \underset{as0}{\Phi} = \underset{as}{\overset{\text{steady}}{\Phi}} + \frac{\underset{as}{\overset{total}{Q}}}{4\pi}[-\ln(4\alpha t)+\gamma] \qquad (4.103)$$

4.6.3 application of area-sink of order one and degree zero.

Transient area-sinks can be used to model the seasonally varying net precipitation or evapotranspiration that is added to or removed from the groundwater. Further applications are the leakage through the bottoms of lakes and rivers or from one aquifer into an other and artificially induced infiltration from drainage fields or irrigation. The potential of an area-sink will be used in chapter 5 to construct a far-field function.

4.6.4 area-sink of order two and degree zero.

For area-sinks of order two the infiltration inside varies linearly with x and y. The first part of the potential is chosen to be equal to

$$\overset{1}{\Phi} = \begin{cases} 0 & \text{outside} \\ -(\underset{0x}{\epsilon}x + \underset{0y}{\epsilon}y + \underset{0}{\epsilon})\alpha t & \text{inside} \end{cases} \tag{4.104}$$

However, in this case not only the potential jumps across the edges but also the normal component of the discharge. The components in x and y direction corresponding to the potential $\overset{1}{\Phi}$ are

$$\overset{1}{Q}_x = -\frac{\partial}{\partial x}\overset{1}{\Phi} = \underset{0x}{\epsilon}\alpha t \tag{4.105}$$

and

$$\overset{1}{Q}_y = -\frac{\partial}{\partial y}\overset{1}{\Phi} = \underset{0y}{\epsilon}\alpha t \tag{4.106}$$

Now $\overset{2}{\Phi}$ has to contain line-sinks of order zero and degree one to cancel the jump in the normal discharge. The discontinuity in the component of the discharge normal to side j is found from the discharge components (4.105) and (4.106) and the orientation θ_j of the side. The strength of the line-sink at that side has the same magnitude and opposite sign

$$\begin{aligned} \sigma_j &= -\,(\overset{1}{Q}_x \sin\theta_j - \overset{1}{Q}_y \cos\theta_j) \\ &= -\,(\underset{0x}{\epsilon}\sin\theta_j - \underset{0y}{\epsilon}\cos\theta_j)\alpha t = \underset{1}{\sigma}_j t \end{aligned} \tag{4.107}$$

The potential (4.68) is used to get the potential for the line-sink at side j

$$\begin{aligned} \underset{ls1}{\Phi}_j = \underset{1}{\sigma}_j\Big[& \frac{1}{4\pi}(\xi_j - \overset{2}{\xi}_j)\{\frac{(\xi_j - \overset{2}{\xi}_j)^2 + \eta_j^2}{12\alpha} + \frac{\eta_j^2}{6\alpha} + t\}\mathrm{E}_1(\frac{(\xi_j - \overset{2}{\xi}_j)^2 + \eta_j^2}{4\alpha t}) \\ & - \frac{1}{4\pi}(\xi_j - \overset{1}{\xi}_j)\{\frac{(\xi_j - \overset{1}{\xi}_j)^2 + \eta_j^2}{12\alpha} + \frac{\eta_j^2}{6\alpha} + t\}\mathrm{E}_1(\frac{(\xi_j - \overset{1}{\xi}_j)^2 + \eta_j^2}{4\alpha t}) \\ & + \frac{\eta_j}{2\sqrt{\pi}}\{\frac{\eta_j^2}{6\alpha} + t\}\int_{\frac{\eta_j}{2\sqrt{\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi_j - \overset{2}{\xi}_j}{\eta_j}) - \mathrm{erfc}(u\frac{\xi_j - \overset{1}{\xi}_j}{\eta_j})\}\,du \\ & - \frac{\sqrt{\alpha t}}{3\sqrt{\pi}}\{\frac{\eta_j^2}{4\alpha} + t\}e^{-\frac{\eta_j^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi_j - \overset{2}{\xi}_j}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi_j - \overset{1}{\xi}_j}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\ & - \frac{t}{12\pi}\{(\xi_j - \overset{2}{\xi}_j)e^{-\frac{(\xi_j - \overset{2}{\xi}_j)^2 + \eta_j^2}{4\alpha t}} - (\xi_j - \overset{1}{\xi}_j)e^{-\frac{(\xi_j - \overset{1}{\xi}_j)^2 + \eta_j^2}{4\alpha t}}\}\Big] \end{aligned} \tag{4.108}$$

where the strength σ_j is given by (4.107). Line-doublets of order one and degree one are needed to cancel the the jump in potential. The global coordinates along side j are given by

$$x = \overset{1}{x}_j + (\xi_j - \overset{1}{\xi}_j)\cos\theta_j \qquad \overset{1}{\xi}_j \le \xi_j \le \overset{2}{\xi}_j \tag{4.109}$$

$$y = \overset{1}{y}_j + (\xi_j - \overset{1}{\xi}_j)\sin\theta_j \qquad \overset{1}{\xi}_j \le \xi_j \le \overset{2}{\xi}_j \tag{4.110}$$

The strength of the line-doublet along side j is obtained from the jump in the potential (4.104)

$$\begin{aligned}
\lambda_j =&(\underset{0x}{\epsilon}x + \underset{0y}{\epsilon}y + \underset{0}{\epsilon})\alpha t\\
=&\{\underset{0x}{\epsilon}(\overset{1}{x}_j + (\xi_j - \overset{1}{\xi}_j)\cos\theta_j) + \underset{0y}{\epsilon}(\overset{1}{y}_j + (\xi_j - \overset{1}{\xi}_j)\sin\theta_j) + \underset{0}{\epsilon}\}\alpha t\\
=&\big[\{\underset{0x}{\epsilon}(\overset{1}{x}_j - \overset{1}{\xi}_j\cos\theta_j) + \underset{0y}{\epsilon}(\overset{1}{y}_j - \overset{1}{\xi}_j\sin\theta_j) + \underset{0}{\epsilon})\}\alpha t\big] + \xi_j\big[(\underset{0x}{\epsilon}\cos\theta_j + \underset{0y}{\epsilon}\sin\theta_j)\alpha t\big]\\
=&t\underset{c1}{\lambda}_j + \xi_j t\underset{l1}{\lambda}_j \qquad \overset{1}{\xi}_j \le \xi_j \le \overset{2}{\xi}_j
\end{aligned} \tag{4.111}$$

The potential (4.89) is used to get the potential for the line-doublet at side j

$$\begin{aligned}
\underset{db1}{\Phi}_j = (\underset{c1}{\lambda}_j + \xi_j\underset{l1}{\lambda}_j)[&+ \frac{\eta_j}{8\pi\alpha}\{(\xi_j - \overset{2}{\xi}_j)\mathrm{E}_1(\frac{(\xi_j - \overset{2}{\xi}_j)^2 + \eta_j^2}{4\alpha t}) - (\xi_j - \overset{1}{\xi}_j)\mathrm{E}_1(\frac{(\xi_j - \overset{1}{\xi}_j)^2 + \eta_j^2}{4\alpha t})\}\\
&+ \frac{1}{2\sqrt{\pi}}\{\frac{\eta_j^2}{2\alpha} + t\}\int_{\frac{\eta_j}{2\sqrt{\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi_j - \overset{2}{\xi}_j)}{\eta_j}) - \mathrm{erfc}(u\frac{\xi_j - \overset{1}{\xi}_j)}{\eta_j})\}\,du\\
&- \frac{\sqrt{t}\eta_j}{4\sqrt{\pi}\sqrt{\alpha}}e^{-\frac{\eta_j^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi_j - \overset{2}{\xi}_j}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi_j - \overset{1}{\xi}_j}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}]\\
+\underset{l1}{\lambda}_j[&- \frac{\eta_j}{4\pi}\{\{\frac{(\xi_j - \overset{2}{\xi}_j)^2 + \eta_j^2}{4\alpha} + t\}\mathrm{E}_1(\frac{(\xi_j - \overset{2}{\xi}_j)^2 + \eta_j^2}{4\alpha t})\\
&- \{\frac{(\xi_j - \overset{1}{\xi}_j)^2 + \eta_j^2}{4\alpha} + t\}\mathrm{E}_1(\frac{(\xi_j - \overset{1}{\xi}_j)^2 + \eta_j^2}{4\alpha t})\}\\
&+ \frac{t\eta_j}{4\pi}\{e^{-\frac{(\xi_j - \overset{2}{\xi}_j)^2 + \eta_j^2}{4\alpha t}} - e^{-\frac{(\xi_j - \overset{1}{\xi}_j)^2 + \eta_j^2}{4\alpha t}}\}]
\end{aligned} \tag{4.112}$$

where the strengths $\underset{c1}{\lambda}_j$ and $\underset{l1}{\lambda}_j$ can be obtained from (4.111)

The potential $\overset{2}{\Phi}$ is equal to the sum of the potentials for the line-sinks (4.108) and the line-doublets (4.112)

$$\overset{2}{\Phi} = \sum_{j=1}^{n}[\underset{ls1}{\Phi}_j + \underset{db1}{\Phi}_j] \tag{4.113}$$

And the total potential for an area-sink of order one and degree one with a boundary polygon of n sides is given by the sum of (4.104) and (4.113)

$$\underset{as1}{\Phi} = \begin{cases} \sum_{j=1}^{n}[\underset{ls1}{\Phi}_j + \underset{db1}{\Phi}_j] & \text{outside}\\ -(\underset{0x}{\epsilon}x + \underset{0y}{\epsilon}y + \underset{0}{\epsilon})\alpha t + \sum_{j=1}^{n}[\underset{ls1}{\Phi}_j + \underset{db1}{\Phi}_j] & \text{inside} \end{cases} \tag{4.114}$$

4.6.5 checking area-sink of order two and degree zero.

4.6.5.1 continuity. The function for the area-sink was constructed in such a way that both the potential and the discharge are continuous across the boundary of the area-sink so that the change from one differential equation to an other does not cause physical discontinuities.

4.6.5.2 initial condition. The potential is zero at time zero for all parts, so that the sum is equal to zero too.

4.6.5.3 boundary condition. The element has been constructed in such a way that the potential and the normal discharge are continuous. All parts of the potential fulfill the boundary condition at infinity so that the sum also has zero discharges at infinity.

4.6.6 area-sinks of degree higher than zero.

It is possible to create an area-sink of order one and degree one in the same fashion. For the first part of the potential the following function is chosen in stead of (4.96)

$$\overset{1}{\Phi} = \begin{cases} 0 & \text{outside} \\ -\underset{1}{\epsilon}\frac{\alpha t^2}{2} & \text{inside} \end{cases} \tag{4.115}$$

The the line-doublets of order zero and degree one in (4.97) are replaced by line-doublets of order zero and degree two with strength $\lambda_{c2} = \underset{1}{\epsilon}\frac{\alpha}{2}t^2$. For higher degrees one needs a higher power of t in (4.115) and higher degree line-doublets. However the line-doublets of degree higher than one were not derived, what needs to be done before these area-sinks can be used.

In a similar way area-sinks of degree higher than one can be derived from (4.104) and (4.113).

5. final steady state and far-field functions

If the conditions, that are imposed on transient groundwater flow, stop changing in time, then the flow will gradually vary less in time and will converge to steady state. This steady state will be referred to in this thesis as the final steady state. It is called final to distinguish this steady state from the steady state used as the initial condition for the transient flow problem. The name initial will be used for the latter steady state.

In many problems of transient flow there is no final steady state, for example if seasonal changes in regional groundwater flow are modeled. However, there are cases in which the final steady state is important, like the transition from an existing situation to a planned future situation. The construction of a canal can have a major impact on the regional groundwater flow (e.g. the Tennessee-Tombigbee Waterway in which the water level is much lower than the original groundwater table). In such cases, one is not only interested in the groundwater flow in the future situation, but also in the time it takes before this new situation has established itself.

Besides the practical importance of proper convergence to a final steady state, there is a theoretical need for it. If a method can not model proper physical behavior, then the reliability and usefulness of the method are limited for practical modeling.

The convergence to a final steady state corresponds to the strong final condition that was introduced in chapter 4, equation (4.4).

If the potential for a transient element fulfills the strong final condition by itself then that potential converges to the potential for the corresponding steady element. The concern for the final steady state stems from the fact that not all transient elements, that were derived in chapter 4, fulfill the strong final condition. It was replaced by a weaker condition (4.5) for the well, line-sink and area-sink. Under the weak condition the potential for a transient element converges to the potential for the corresponding steady element plus a function of the time.

When the weak final condition was allowed for individual elements in stead of the strong condition, it was mentioned that the sum of the elements in a model should still fulfill the strong final condition (4.4).

Therefore a potential will be examined which is equal to the sum of potentials for transient elements. Using (4.27), (4.61), (4.84) and (4.103), the potential for a single element at large values of the time can be written as

$$\underset{j}{\Phi} = \overset{\text{steady}}{\underset{j}{\Phi}}\Big|_{R=L} + \frac{\overset{total}{\underset{j}{Q}}}{4\pi}[-\ln(\frac{4\alpha t}{L^2}) + \gamma] \qquad (t \to \infty) \tag{5.1}$$

where j is the number of the element. The potential of the steady element is normalized with respect to the length L (see chapter 3). The limit for $t \to \infty$ of the sum of m potentials of transient elements is equal to

$$\begin{aligned}\lim_{t\to\infty}\sum_{j=1}^{m}\underset{j}{\Phi} &= \sum_{j=1}^{m}\Big\{\overset{\text{steady}}{\underset{j}{\Phi}}\Big|_{R=L} + \frac{\overset{total}{\underset{j}{Q}}}{4\pi}[-\ln(\frac{4\alpha t}{L^2}) + \gamma]\Big\} \\ &= \sum_{j=1}^{m}\overset{\text{steady}}{\underset{j}{\Phi}}\Big|_{R=L} + \frac{\sum_{j=1}^{m}\overset{total}{\underset{j}{Q}}}{4\pi}[-\ln(\frac{4\alpha t}{L^2}) + \gamma] \qquad (t \to \infty)\end{aligned} \tag{5.2}$$

Application of the strong final condition tells that the sum of the transient potentials has to be equal to the sum of the corresponding steady potentials so that the sum of the discharges has to be equal to zero

$$\sum_{j=1}^{m} \overset{total}{Q}_{j} = 0 \tag{5.3}$$

and equation (5.2) becomes

$$\lim_{t\to\infty} \sum_{j=1}^{m} \Phi_j = \sum_{j=1}^{m} \overset{steady}{\Phi}_j \Big|_{R=L} \tag{5.4}$$

Comparison of this result with (3.32) shows that (5.4) corresponds to a constant $\mathcal{C}$ that is equal to zero. Using (3.35) also, it is possible to say that the sum of the potentials of transient elements either converges to a steady potential with $\mathcal{C} = 0$ and $\mathcal{Q} = 0$

$$\lim_{t\to\infty} \sum_{j=1}^{m} \Phi_j = \overset{steady}{\Phi} \Big|_{\substack{\mathcal{C}=0 \\ \mathcal{Q}=0}} \tag{5.5}$$

or it increases logarithmically with time for $t \to \infty$ (compare (5.2)).

A transient model consists of the potential for transient elements super imposed on the potential for the initial steady state. In an application it is very hard to meet the condition (5.3) that the sum of the discharges of the transient elements is equal to zero. The discharges of the elements are not all known beforehand in modeling of regional groundwater flow. Usually the strengths for a large fraction of the of elements have to be determined from the condition that the head has a specified value at the element.

example.

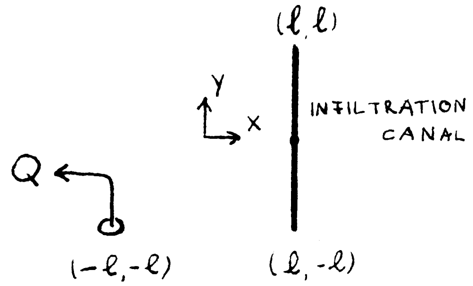

figure 5.1. *Well and infiltration canal*

The simple case of a well near an infiltration canal is used as an example (see figure 5.1) throughout this chapter. The aquifer is confined, with thickness H, hydraulic conductivity k, and diffusivity α. Initially the head is constant and there is no flow. The well (4.16) starts pumping a constant discharge Q at time $t = 0$. The infiltration canal is represented as a line-segment with a distributed discharge, which is constant along the segment, and piecewise constant in time. This is modeled by line-sinks of order one and degree zero (4.56) at the line-segment with different starting times. The strengths σ of the line-sinks are calculated from the condition that the head at the center of the line-segment is equal to the initial head at all times.

The total strength along the line-segment is plotted versus the time on a logarithmic scale in figure 5.2. The total strength does not converge to a constant. The model does not give a final steady state.

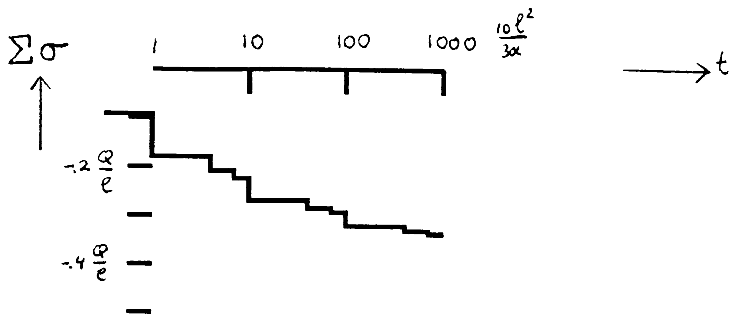

figure 5.2. *Discharge of canal as function of time*

There is no degree of freedom that can be used to satisfy the condition (5.3) that the sum of the discharges of the transient elements becomes constant for large values of the time. Each element in the potential represents an aquifer feature, and the degrees of freedom are determined by the specific conditions associated with the aquifer features, which the elements are representing. Therefor the potential may not converge to a steady state in the entire model. This is not necessarily a problem. The purpose of the model is to get information about the area of interest. If the flow in the area of interest converges properly to the final steady state then the model serves its purpose.

As with the analytic element method for steady flow (see chapter 3), a transient analytic element model consists of an area of interest, a near-field and a far-field. The area of interest was modeled minutely, the elements became gradually coarser in the near-field until the far-field was reached that does not contain any elements at all. The near-field was extended in steps until the influence of the elements added in the last step on the area of interest was negligible. That also meant that all the aquifer features had been included that had a significant influence on the area of interest.

If a transient model is set up in the same way, so that all aquifer features, that are important for the flow in the area of interest, are modeled explicitly by elements in the near-field, then the flow in the area of interest converges to steady flow. The model outside of the area of interest gradually lacks more controlling aquifer features as the distance to the area of interest increases; the model will gradually deviate more from the physical reality.

example.

To illustrate this, the case of a well and an infiltration canal is used again. With only the well and the canal the model did not even converge to a steady state in the area of interest (see figure 5.2). Control on the flow in the area of interest is added in the form of two sections of nearby rivers, which are shown in figure 5.3. The river-sections are modeled each by one element, in the same way as the canal, and the head at the centers is also kept at the initial level.

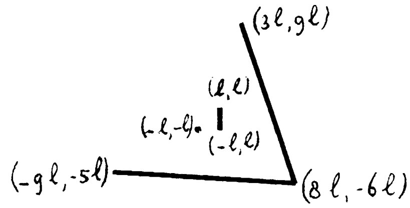

figure 5.3. *Well and infiltration canal with sections of nearby rivers*

With the extended near-field the distributed discharge of the canal converges to a constant value, as can be seen in figure 5.4. However, the model does only become steady in the area of interest. That the heads keep changing away from the center of the model is shown in figure 5.5 by means of contour plots of the head at two different times.

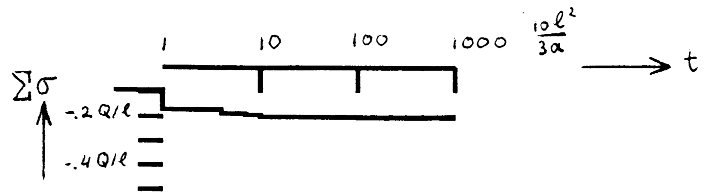

figure 5.4. *Discharge of canal as function of time*

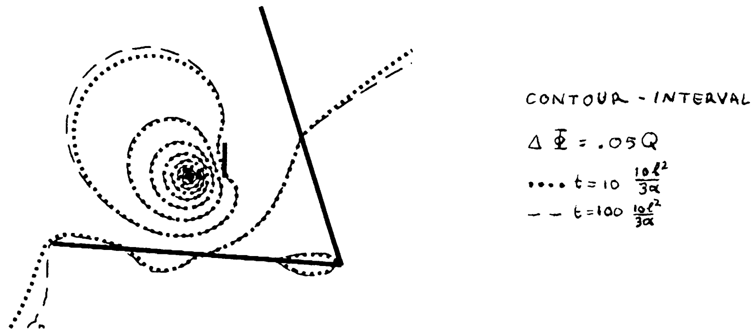

figure 5.5. *Equipotentials at different times for well and infiltration canal*

If the near-field is not extended far enough to include all aquifer features that have a significant influence on the flow in the area of interest, then the model will not becomes steady there. Models that are too small are used as simple problems to test a computer program and in the initial phase of a project, when the problem is explored.

If the near-field is too small, then extra control has to be added to the model in order to make the area of interest converge to a final steady state. This may be done by adding special transient functions to the potential that change the parameters of the steady state in (5.5). Two of these special functions will be derived later in this chapter. The one function gives a value unequal to zero for the constant $\mathcal{C}$ in the limit $t \to \infty$. The other makes it possible for a transient potential that does not fulfill (5.3) to converge to a steady state. Thus the final steady state can have a value of the sum of the total discharges $\mathcal{Q}$ that is different from the initial steady state. These functions will be called transient far-field functions since they change the values of $\mathcal{C}$ and $\mathcal{Q}$, which are the parameters of the steady far-field (3.34). To be able to use these functions the final steady state needs to be determined so that the amount of change of these parameters is known.

final steady state

A steady state can be calculated directly with the conditions and elements that are used in the transient model, without using transient elements in the limit for $t \to \infty$. This steady state is not necessarily the final steady state of the transient model. Far-field functions with the correct values of the changes in $\mathcal{C}$ and $\mathcal{Q}$ have to be used in order for the entire model to converge to the calculated steady state.

This steady state is modeled using the conditions and corresponding elements from the transient state and the elements from the initial steady state. The strengths of the elements from the initial steady state are all known. There is a condition for every unknown strength of the new elements. There is one more unknown in the model however, the constant. The corresponding condition usually is the reference equation. In this case a steady state is being determined that is connected to the initial steady state. Thus it is more logical to select a condition that connects the far-fields of the initial and the final steady state, in stead of adopting some arbitrary reference point and - head. One choice is to set the total discharge of the new elements equal to zero, so that the sum of the discharges of the elements in the steady state that is being calculated is equal to the sum of the discharges in the intial steady state

$$\underset{f}{Q} = \underset{o}{Q} \tag{5.6}$$

where the subscripts $_f$ and $_o$ indicate final and initial state respectively. Another choice is that the values of the constants in the initial and final steady states are equal

$$\underset{f}{\mathcal{C}} = \underset{o}{\mathcal{C}} \tag{5.7}$$

For practical purposes one can determine the final steady state twice, once with each condition. If the two are close in the area of interest the area of interest is apparently not significantly influenced by the far-field, so that all aquifer features that control the flow there are included in the near-field. Since there is enough control on the flow in the area of interest included in the model, the transient state will converge to the same values in the area of interest.

example.

The final steady state will be determined for the cases with a well and infiltration canal. Both the cases without and with the river-sections will be given. The steady model for the case with only the well and the canal shown in figure 5.1 is set up with a steady well of discharge Q (3.4) and a steady line-sink of order one (3.13) with a specified head at the center equal to the head that was specified at the center of the canal in the transient model (the initial head in this example). The model for the final steady state is completed by either the condition that the sum of the discharges is equal to zero (the value in the initial steady state), or that the constant $\mathcal{C}$ has the same value. Contour plots of the heads for the two solutions obtained with the two different conditions are given in figure 5.6.

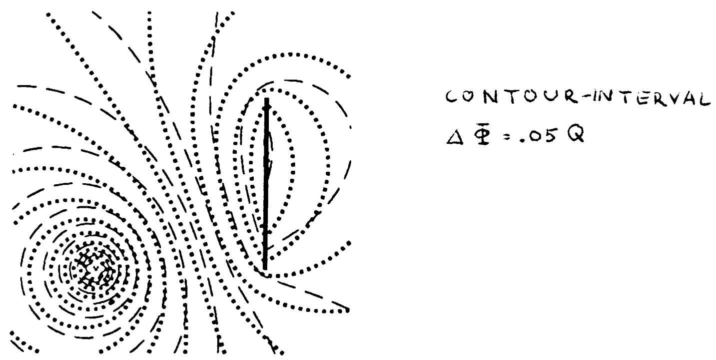

figure 5.6. *Equipotentials for two steady states with well and infiltration canal*

The lines of constant head and the discharges of the canal are quite different, indicating that the model does not have a large enough near-field, as was seen from the transient calculations also (see figure 5.2). In the transient calculations for the case in which the river-sections were added, the control on the flow in the area of interest was large enough to give a steady state locally. That the near-field is extended far enough can also be seen from the comparison of the solutions for the final steady state with the two different conditions (see figure 5.7)

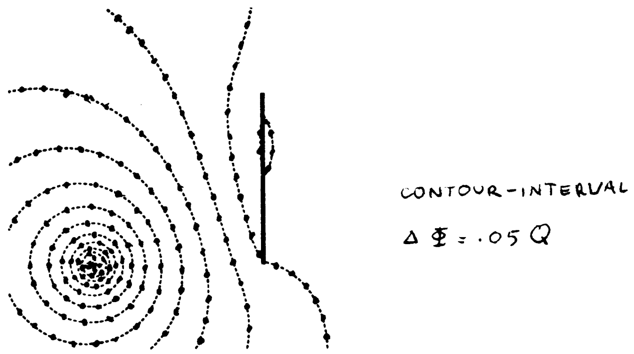

figure 5.7. *Equipotentials for two steady states with well and infiltration canal plus river-sections*

If the two solutions differ too much in the area of interest, then the model is not extended far enough outward. In that case the far-field has to simulate the influence of the important aquifer features that are left out of the model. The special transient functions then have to be added to the potential for the transient state and the parameters have to be chosen such that the far-field functions model the influence of the features that are left out.

The special transient functions also have to be used to get the entire model to converge to a final steady state and not just in the area of interest.

The change in $\mathcal{C}$ and $\mathcal{Q}$ that the transient far-field functions have to create can be determined from the far-fields of the initial and final steady state. The equation for the far-field of a steady state (3.34) is recalled

$$\overset{\text{steady}}{\Phi} = \mathcal{C} + \frac{\mathcal{Q}}{4\pi}\ln\left(\frac{x^2+y^2}{L^2}\right)\left(\sqrt{x^2+y^2} \to \infty\right) \tag{5.8}$$

From the vizualization of the imaginary boundary in the far-field, a circle with radius L, potential $\mathcal{C}$ and normal discharge $-\frac{\mathcal{Q}}{2\pi L}$ (see figure 3.9), a change in the constant $\mathcal{C}$ can be interpreted as a general change in the water table that is caused by the aquifer features left out of the model. These aquifer features recharged or discharged an amount of water for a limited time. A different value of $\mathcal{Q}$ means that the continuous recharge from the aquifer features that were not included to the near-field changes.

Two transient functions will be derived next, that effectuate such changes. The first function changes the value of the constant $\mathcal{C}$, by changing the value of the potential evenly in the far-field. The second function produces extra flow to the near-field continuously and so changes the value of $\mathcal{Q}$.

changing $\mathcal{C}$, the constant

A transient far-field function that changes the value of $\mathcal{C}$ changes the level of the potential but does not influence the discharge vector far away. A function that changes the value of the potential uniformly in the far-field has this behavior. Such a function is the potential uniform evapotranspiration. However the extraction should only take place in the far-field. Therefore a function is chosen with a potential $\underset{p}{\Phi}$, which gives extraction outside of the near-field only (see figure 5.8). The extraction lasts for a limited period of time.

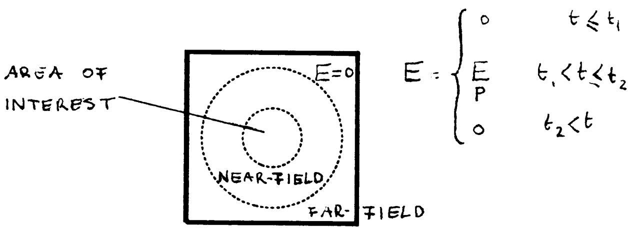

figure 5.8. *Extraction of far-field function for changing the constant*

A potential for extraction away from the near-field is obtained from superposition of the potential of an area-sink of order one and degree zero (4.98) onto the potential for uniform evapotranspiration of order one and degree zero (equation (4.10) with $n = 0$). The strength of the area-sink is equal in magnitude and opposite in sign to the strength of the evapotranspiration so that the extraction of the latter is canceled inside the area-sink. The area-sink is placed in such a way that all the elements included in the model are inside it, so that the far-field function does not disturb extraction conditions in the near-field.

The first part of the potential starts the extraction at time t_{p1}

$$\underset{p1}{\Phi} = \begin{cases} 0 & t \leq t_{p1} \\ -E_p\alpha(t - t_{p1}) + \Phi_{as}|_{\substack{e_0 = -E_p \\ t_0 = t_{p1}}} & t > t_{p1} \end{cases} \tag{5.9}$$

After time t_{p2} the extraction is stopped by the second part of the potential that contains an extraction that cancels the extraction of the first part

$$\underset{p2}{\Phi} = \begin{cases} 0 & t \leq t_{p2} \\ -(-E_p)\alpha(t - t_{p2}) + \Phi_{as}|_{\substack{e_0 = E_p \\ t_0 = t_{p2}}} & t > t_{p2} \end{cases} \tag{5.10}$$

so that the complete potential for the second function to change the value of the potential at infinity is equal to

$$\underset{p}{\Phi} = \begin{cases} 0 & t \leq t_{p1} \\ -E_p\alpha(t - t_{p1}) + \Phi_{as}|_{\substack{e_0 = -E_p \\ t_0 = t_{p1}}} & t_{p1} < t \leq t_{p2} \\ -E_p\alpha(t_{p2} - t_{p1}) + \Phi_{as}|_{\substack{e_0 = -E_p \\ t_0 = t_{p1}}} + \Phi_{as}|_{\substack{e_0 = E_p \\ t_0 = t_{p2}}} & t > t_{p2} \end{cases} \tag{5.11}$$

see figure 5.9.

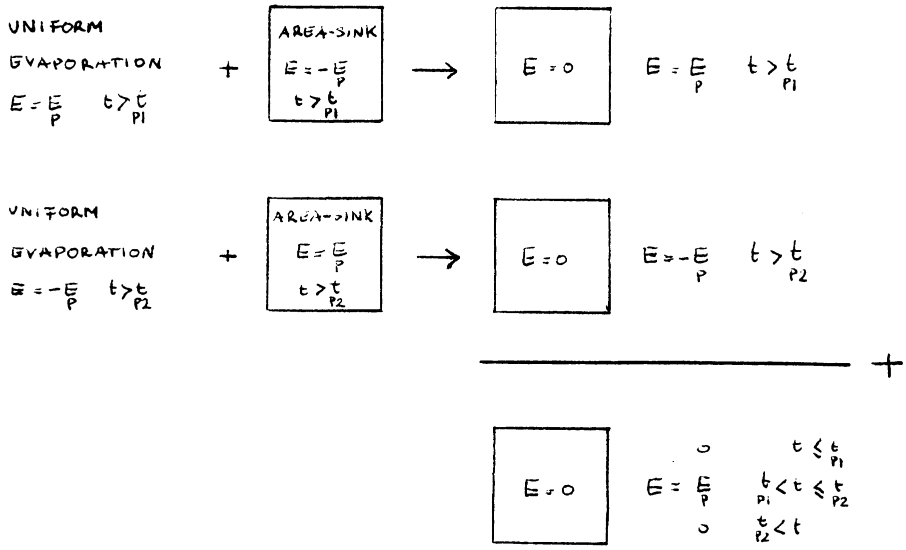

figure 5.9. *Construction of far-field function for changing* Φ_c

The limit for large times of the potential (5.11) is equal to

$$\lim_{t \to \infty} \Phi_p = -E_p \alpha (t_{p2} - t_{p1}) \tag{5.12}$$

since the two area-sinks cancel each other in the limit $t \to \infty$. The limit (5.12) has one constant value in the entire plane, so that the condition (5.7) can be replaced by

$$\mathcal{C}_f = \mathcal{C}_o + \lim_{t \to \infty} \Phi_p \tag{5.13}$$

where the subscript f indicates the final steady state. The potential (5.11) has three degrees of freedom, E_p, t_{p1} and t_{p2}. Equation (5.13) gives only one condition, so that two of the three parameters can be chosen freely. The choice is made such that the influence of the far-field in the model is close to the influence of the aquifer features that have not been included in the model explicitly in the form of elements.

example.

In figure 5.6 a steady state has been shown with $Q_f = Q_o$ for the case of just a well and an infiltration canal (without the river-sections). The parameters of the far-field function (5.22) are chosen to create the corresponding amount of change in $\mathcal{C}$. Three different sets of parameters are fiven in figure 5.10. In the same figure the potentials of the farfield functions are shown at the center of the line-segment, where the condition for the specified head is applied.

The first and the second set differ in the duration of the extraction in the far-field. The potential at the center of the line-segment increases more gradually in the beginning for the second set. Later the potentials change similarly so that the second curve has a time lag after the first one. The third set of parameters for the far-field function has the same extraction as the first, but the sides of the area-sink are twice as long. This causes the potential to change more slowly and converge much later to the final value.

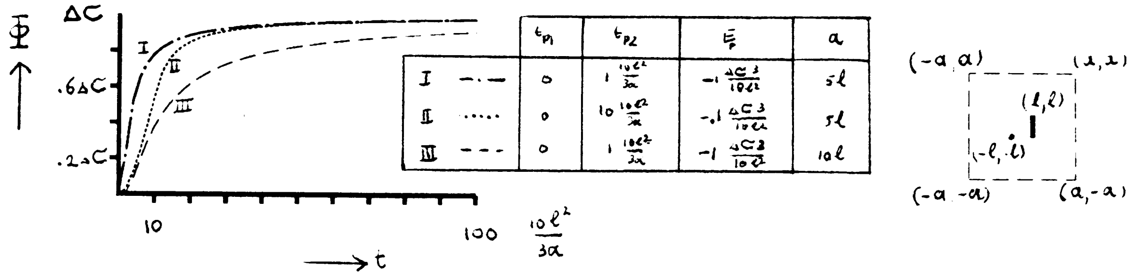

figure 5.10. *Well and canal with far-field function for for C*

For all three sets the flow converges to a steady state, as was to be expected. The distributed discharges of the canal are shown in figure 5.11 (compare figure 5.2 where no far-field function was used). The rates at which the final value are approached differ in the same fashion as those for the potential for the far-field function.

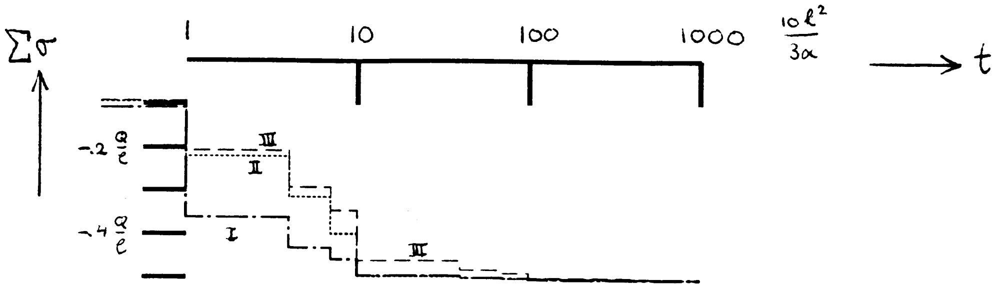

figure 5.11. *Discharge of canal as function of time with far-field function for C*

The parameters for the far-field function for changing C have to be chosen such that the product of the extraction E_p and the difference of the times $t_{p2} - t_{p1}$ is equal to the value that follows from the required change of C (see equations (5.12) and (5.13)). The locical choice for the time t_{p1} is the starting time of the transient elements, so that the transient effects in the far-field and the near-field start simultaneously. This leaves the size of the area-sink and one of the parameters t_{p2} and E_p to be chosen. The smaller the duration $t_{p2} - t_{p1}$ of the extraction is chosen, the more sharply the increase in potential is and the more it varies between different locations. Increasing the size of the area-sink counteracts the latter effect: the larger the area-sink, the more evenly the potential changes in the near-field, but the slower the rate of convergence becomes.

changing Q, the discharge at infinity

A transient far-field function that changes the value of Q changes the flow toward the near-field but does not change the constant. A transient ring-source with a large diameter has this effect. The ring-source injects water into the aquifer evenly along a circle (see figure 5.12).

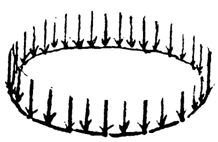

figure 5.12. *Ring-source*

The potential for a ring-source will be obtained by superposition of two solutions. The first one is a solution presented by Glover(1974) and the second one is a well of degree zero (4.16).

Glover(1974) presented the solution for a transient well of degree zero in the center of a circular island (see figure 5.13). Initially the head is constant. At time zero the well starts pumping. The piezometric head stays at the initial level on a circle with radius R around the well

$$\underset{g}{\Phi}\Big|_{r=L} = 0 \tag{5.14}$$

where the subscript $_g$ stands for Glover's solution and r is the radial coordinate measured from the center of the island.

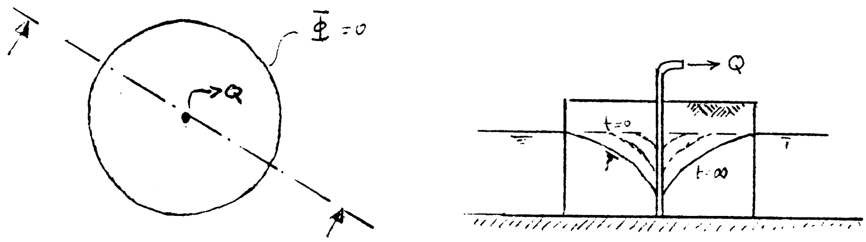

figure 5.13. *Solution of Glover(1974)*

Glover obtained his solution to the boundary value problem as a Fourier series on the disk $r < L$, so that it cannot be used for $r > L$. The solution that he presented is

$$\Phi_g = \frac{Q}{2\pi}[\ln\frac{r}{L} + \sum_{n=1}^{\infty}\frac{2J_0(\beta_n r)e^{-\beta_n^2 \alpha t}}{(\beta_n L)^2 J_1^2(\beta_n L)}] \qquad r < L \tag{5.15}$$

where the factors β_n correspond to the zeros of the Bessel function of the first kind of order zero

$$J_0(\beta_n L) = 0 \tag{5.16}$$

The limit for large times of this potential also is given by Glover(1974)

$$\lim_{t\to\infty}\underset{g}{\Phi} = \frac{Q}{2\pi}\ln\frac{r}{L} \quad r \le L \tag{5.17}$$

The difference between Glover's solution (5.15) and the well of degree zero (4.16) is the condition for Glover's solution that the potential does not change at a distance L away from the well (5.14), while the potential for the well of degree zero remains constant at infinity (4.2).

The solution of Glover consists of a transient well of degree zero at the origin and a discharge at a circle of radius L around the well, that maintains the potential on the circle at a constant level. If the potential for a well of degree zero is subtracted from the potential for Glover's solution, then the potential due to the discharge at the circle remains. This is the potential for a ring-source $\underset{r}{\Phi}$, so that the potential $\underset{r}{\Phi}$ is equal to the sum of the potentials (5.15) and (4.16) with opposite discharges

$$\begin{aligned}\underset{r}{\Phi} &= \underset{g}{\Phi} - \underset{w0}{\Phi} \\ &= \frac{\Xi}{2\pi}[\ln\frac{r}{L} + \sum_{n=1}^{\infty}\frac{2J_0(\beta_n r)e^{-\beta_n^2\alpha t}}{(\beta_n L)^2 J_1^2(\beta_n L)}] - \frac{-\Xi}{4\pi}\mathrm{E}_1(\frac{r^2}{4\alpha t}) \quad r \le L\end{aligned} \tag{5.18}$$

where the discharge of Glover's solution is equal to Ξ and the discharge of the well of degree zero is equal to $-\Xi$. The potential (5.18) can be written as

$$\underset{r}{\Phi} = \frac{\Xi}{4\pi}[\ln \frac{r^2}{L^2} + \sum_{n=1}^{\infty} \frac{4J_0(\beta_n r)e^{-\beta_n^2 \alpha t}}{(\beta_n L)^2 J_1^2(\beta_n L)} + \mathrm{E}_1(\frac{r^2}{4\alpha t})] \quad r \leq L \tag{5.19}$$

In order to show that this function can be used for the convergence to a final steady state with a new value of $\mathcal{Q}$, the behavior for large values of time is considered. The limit for $t \to \infty$ of the potential of the ring-source (5.19) is found using (5.17) and (4.27)

$$\begin{aligned} \lim_{t\to\infty} \underset{r}{\Phi} &= \lim_{t\to\infty} (\underset{g}{\Phi} - \underset{w0}{\Phi}) \\ &= \frac{\Xi}{4\pi} \ln \frac{r^2}{L^2} + \frac{-\Xi}{4\pi}(\ln \frac{r^2}{L^2} - \ln \frac{4\alpha t}{L^2} + \gamma) \\ &= \frac{\Xi}{4\pi}(\ln \frac{4\alpha t}{L^2} - \gamma) \qquad\qquad r \leq L \end{aligned} \tag{5.20}$$

Combination of the result (5.19) with the sum (5.2) of potentials at large times for transient elements gives

$$\begin{aligned} \lim_{t\to\infty} \sum_{j=1}^{m} \underset{j}{\Phi} + \underset{r}{\Phi} &= \sum_{j=1}^{m} \overset{\text{steady}}{\underset{j}{\Phi}} |_{R=L} + \frac{\sum_{j=1}^{m} \overset{total}{\underset{j}{Q}}}{4\pi}[-\ln(\frac{4\alpha t}{L^2}) + \gamma] + \frac{\Xi}{4\pi}(\ln \frac{4\alpha t}{L^2} - \gamma) \\ &= \sum_{j=1}^{m} \overset{\text{steady}}{\underset{j}{\Phi}} |_{R=L} + \frac{\Xi - \sum_{j=1}^{m} \overset{total}{\underset{j}{Q}}}{4\pi}(\ln \frac{4\alpha t}{L^2} - \gamma) \end{aligned} \tag{5.21}$$

The limit exists only if

$$\Xi = \sum_{j=1}^{m} \underset{j}{Q} \tag{5.22}$$

Thus the ring-source indeed changes the sum $\mathcal{Q}$ of the total discharges $\mathcal{Q}$. The parameter Ξ is the amount of the change. In this case the limit (5.21) is equal to the sum of the normalized steady elements that correspond to the transient elements, so that the constant $\mathcal{C}$ is not influenced (compare (3.32) and (5.4)) and the function $\underset{r}{\Phi}$ is a function that changes only the discharge at infinity.

example.

In figure 5.6 a steady state has been shown with $\underset{f}{\mathcal{C}} = \underset{o}{\mathcal{C}}$ for the case of just a well and an infiltration canal (without the river-sections). The far-field function for changing $\mathcal{Q}$ (5.19) is added to the potential with the corresponding amount of change in $\mathcal{Q}$ (see figure 5.14).

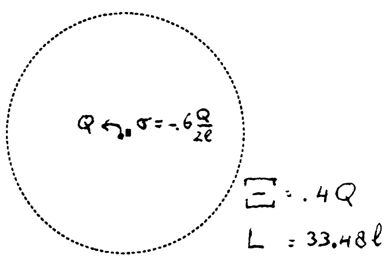

figure 5.14. *Well and canal with far-field function for $\mathcal{Q}$*

Now the distributed discharge of the canal converges to a constant value, as can be seen in figure 5.15 (compare figure 5.2 where no far-field function was used).

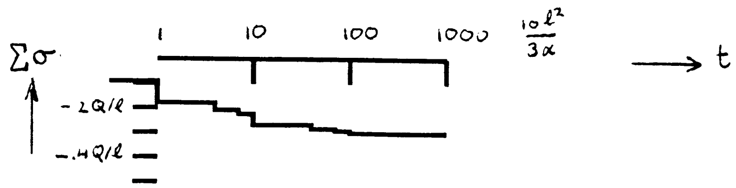

figure 5.15. *Discharge of canal as function of time with far-field function for Q*

The potential of the ring-source (5.19) is not valid for values of the radial coordinate r that are larger than L. This does not complicate practical modeling, but is rather unelegant in a model that only contains functions that are defined for radii up to infinity.

A real drawback of the function is the formulation as a series in which each term contains trancendental functions, which make the computation time consuming.

The logical choice for an element for changing the discharge Q at infinity would have been a transient element that injects water at infinity, rather than at some arbitrary distance L. The potential for a transient well at infinity could not be found however.

6. a model with transient elements

In this chapter the implementation of the transient elements in a model for transient groundwater flow is covered. The elements have been derived in chapter 4 and 5. The aquifer parameters, the initial steady state and the transient aquifer features are input to the model. Some features give rise to elements with strengths which are a priori unknown and can be determined from a condition, that is specific for the feature. The potential and the components of the discharge vector are known as functions of location and time, once the unknown strengths have been determined. Using these functions, output can be generated such as contour plots of the head, numeric values of the head, of the discharge vector or of the extraction at a specified location and time, or pathlines, following a water particle downstream or tracing its origin upstream.

The model contains transient wells, line-sinks and area-sinks of degree zero and two far-field functions. The line-sinks and area-sinks have an order one. Besides these elements with the strength specified, line-segments can be entered with a specified head. These line-segments are represented by a number of line-sinks each with a different starting time. The strengths of these line-sinks are a priori unknown, and have to be determined from the condition that the value of the head at the center of the line-segment is equal to the specified value. Thus a system of equations is set up that is solved to obtain the values of the unknown strengths.

Before the equations are discussed, a notation is introduced, which makes it possible to use the potentials for the elements in equations without having to give the entire expression for the potential, while the important parameters are clearly identified in the equation.

notation

well.

Let well number j be located at $(\underset{ul}{x}_j, \underset{ul}{y}_j)$, have a discharge $\underset{0}{Q}_j$ and start at time $\underset{ul}{t}_j$. Define a real function $\underset{ul0}{\Lambda}_j$ for well j using the potential (4.16) for a well of degree zero

$$\begin{aligned} &\underset{ul0}{\Lambda}_j(x,y,t) = 0 && t \leq \underset{ul}{t}_j \\ &\underset{ul0}{\Lambda}_j(x,y,t) = -\frac{1}{4\pi}\mathrm{E}_1\Big(\frac{(x-\underset{ul}{x}_j)^2 + (y-\underset{ul}{y}_j)^2}{4\alpha(t-\underset{ul}{t}_j)}\Big) && t > \underset{ul}{t}_j \end{aligned} \tag{6.1}$$

Using this function, the potential for well j can be written as

$$\underset{0}{Q}_j \underset{ul0}{\Lambda}_j(x,y,t) \tag{6.2}$$

and the potential for all $\underset{ul}{m}$ wells with a given strength as

$$\sum_{j=1}^{\underset{ul}{m}} [\underset{0}{Q}_j \underset{ul0}{\Lambda}_j(x,y,t)] \tag{6.3}$$

line-sink.

Let the strength specified line-sink number j have end-points $(\overset{1}{x}_j, \overset{1}{y}_j)$ and $(\overset{2}{x}_j, \overset{2}{y}_j)$, strength $\underset{0}{\sigma}_j$ and starting time $\underset{ls}{t}_j$. A function $\underset{ls0}{\Lambda}_j$ is defined for line-sink j using the potential (4.56) for a line-sink of order one and degree zero

$$
\begin{aligned}
\underset{ls0}{\Lambda}_j(x,y,t) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \le \underset{ls}{t}_j \\
\underset{ls0}{\Lambda}_j(x,y,t) &= -\frac{\sqrt{\alpha(t-\underset{ls}{t}_j)}}{2\sqrt{\pi}} e^{-\frac{\eta^2}{4\alpha(t-\underset{ls}{t}_j)}} \{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha(t-\underset{ls}{t}_j)}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha(t-\underset{ls}{t}_j)}})\} \\
&+ \frac{1}{4\pi}\{(\xi-\overset{2}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-\underset{ls}{t}_j)}) - (\xi-\overset{1}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-\underset{ls}{t}_j)})\} \\
&+ \frac{\eta}{2\sqrt{\pi}} \int_{\frac{\eta}{2\sqrt{\alpha(t-\underset{ls}{t}_j)}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\, du \qquad t > \underset{ls}{t}_j
\end{aligned}
\tag{6.4}
$$

where ξ, $\overset{1}{\xi}$, $\overset{2}{\xi}$ and η are given below. The local coordinates ξ and η are parallel and normal to the element respectively (see figure 3.4). The relation to the global coordinates x and y is given by (4.64), (4.65) and (4.66)

$$\xi = (x - \frac{\overset{1}{\underset{ls}{x}}_j + \overset{2}{\underset{ls}{x}}_j}{2})\cos\theta + (y - \frac{\overset{1}{\underset{ls}{y}}_j + \overset{2}{\underset{ls}{y}}_j}{2})\sin\theta \tag{6.5}$$

$$\eta = -(x - \frac{\overset{1}{\underset{ls}{x}}_j + \overset{2}{\underset{ls}{x}}_j}{2})\sin\theta + (y - \frac{\overset{1}{\underset{ls}{y}}_j + \overset{2}{\underset{ls}{y}}_j}{2})\cos\theta \tag{6.6}$$

where θ is equal to the orientation of the line-sink

$$\theta = \arctan(\frac{\overset{2}{\underset{ls}{y}}_j - \overset{1}{\underset{ls}{y}}_j}{\overset{2}{\underset{ls}{x}}_j - \overset{1}{\underset{ls}{x}}_j}) \qquad -\pi < \theta \le \pi \tag{6.7}$$

where the range of the arctangent is from $-\pi$ to π. The ξ_j coordinates of the end-points are equal to

$$\overset{1}{\xi} = -\frac{1}{2}\sqrt{(\overset{2}{\underset{ls}{x}}_j - \overset{1}{\underset{ls}{x}}_j)^2 + (\overset{2}{\underset{ls}{y}}_j - \overset{1}{\underset{ls}{y}}_j)^2} \tag{6.8}$$

$$\overset{2}{\xi} = \frac{1}{2}\sqrt{(\overset{2}{\underset{ls}{x}}_j - \overset{1}{\underset{ls}{x}}_j)^2 + (\overset{2}{\underset{ls}{y}}_j - \overset{1}{\underset{ls}{y}}_j)^2} \tag{6.9}$$

where $\sqrt{(\overset{2}{\underset{ls}{x}}_j - \overset{1}{\underset{ls}{x}}_j)^2 + (\overset{2}{\underset{ls}{y}}_j - \overset{1}{\underset{ls}{y}}_j)^2}$ is the length of the element.

With the function (6.4) the potential for line-sink j can be written as

$$\underset{0}{\sigma}_j \underset{ls0}{\Lambda}_j(x,y,t) \tag{6.10}$$

The potential for all $\underset{ls}{m}$ line-sinks with given strength is equal to

$$\sum_{j=1}^{\underset{ls}{m}} [\underset{0}{\sigma}_j \underset{ls0}{\Lambda}_j(x,y,t)] \tag{6.11}$$

area-sink.

Let the quadrilateral area-sink number j have strength $\underset{0}{\epsilon}_j$, starting time $\underset{as}{t}_j$ and corner-points $(\underset{as}{\overset{1}{x}}, \underset{as}{\overset{1}{y}})$, $(\underset{as}{\overset{2}{x}}, \underset{as}{\overset{2}{y}})$, $(\underset{as}{\overset{3}{x}}, \underset{as}{\overset{3}{y}})$ and $(\underset{as}{\overset{4}{x}}, \underset{as}{\overset{4}{y}})$. A function $\underset{as0}{\Lambda}_j$ is defined using the potential for an area-sink of order one and degree zero which is given by (4.98) and (4.97)

$$
\begin{aligned}
\underset{as0}{\Lambda}(x,y,t) &= 0 && t \le \underset{as}{t}_j \\
\underset{as0}{\Lambda}(x,y,t) &= \begin{cases} \overset{2}{\Phi} & \text{outside} \\ -\alpha t + \overset{2}{\Lambda}_j(x,y,t) & \text{inside} \end{cases} && t > \underset{as}{t}_j
\end{aligned}
\tag{6.12}
$$

where $\overset{2}{\Lambda}_j$ is equal to (compare (4.97))

$$
\begin{aligned}
\overset{2}{\Lambda}(x,y,t) = \alpha \sum_{l=1}^{n} \Big[& \frac{\eta_l}{8\pi\alpha}\{(\xi_l - \overset{2}{\xi}_l)\mathrm{E}_1(\frac{(\xi_l - \overset{2}{\xi}_l)^2 + \eta_l^2}{4\alpha(t - \underset{as}{t}_j)}) - (\xi_l - \overset{1}{\xi}_l)\mathrm{E}_1(\frac{(\xi_l - \overset{1}{\xi}_l)^2 + \eta_l^2}{4\alpha(t - \underset{as}{t}_j)})\} \\
& + \frac{1}{2\sqrt{\pi}}\{\frac{\eta_l^2}{2\alpha} + (t - \underset{as}{t}_j)\} \int_{\frac{\eta_l^2}{2\sqrt{\alpha(t-\underset{as}{t}_j)}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi_l - \overset{2}{\xi}_l}{\eta_l}) - \mathrm{erfc}(u\frac{\xi_l - \overset{1}{\xi}_l}{\eta_l})\}\,du \\
& - \frac{\sqrt{(t - \underset{as}{t}_j)}\eta_l}{4\sqrt{\pi}\sqrt{\alpha}} e^{-\frac{\eta_l^2}{4\alpha(t-\underset{as}{t}_j)}}\{\mathrm{erfc}(\frac{\xi_l - \overset{2}{\xi}_l}{2\sqrt{\alpha(t - \underset{as}{t}_j)}}) - \mathrm{erfc}(\frac{\xi_l - \overset{1}{\xi}_l}{2\sqrt{\alpha(t - \underset{as}{t}_j)}})\}\Big]
\end{aligned}
\tag{6.13}
$$

where $\overset{1}{\xi}_l$, $\overset{2}{\xi}_l$, ξ_l and η_l are given below with $\underset{as}{\overset{5}{x}}_j = \underset{as}{\overset{1}{x}}_j$ and $\underset{as}{\overset{5}{y}}_j = \underset{as}{\overset{1}{y}}_j$

$$
\overset{1}{\xi}_l = -\frac{1}{2}\sqrt{(\underset{as}{\overset{l+1}{x}}_j - \underset{as}{\overset{l}{x}}_j)^2 + (\underset{as}{\overset{l+1}{y}}_j - \underset{as}{\overset{l}{y}}_j)^2} \qquad l = 1,2,3,4 \tag{6.14}
$$

$$
\overset{2}{\xi}_l = \frac{1}{2}\sqrt{(\underset{as}{\overset{l+1}{x}}_j - \underset{as}{\overset{l}{x}}_j)^2 + (\underset{as}{\overset{l+1}{y}}_j - \underset{as}{\overset{l}{y}}_j)^2} \qquad l = 1,2,3,4 \tag{6.15}
$$

$$
\xi_l = (x - \frac{\underset{as}{\overset{l}{x}}_j + \underset{as}{\overset{l+1}{x}}_j}{2})\cos\theta_l + (y - \frac{\underset{as}{\overset{l}{y}}_j + \underset{as}{\overset{l+1}{y}}_j}{2})\sin\theta_l \qquad l = 1,2,3,4 \tag{6.16}
$$

$$
\eta_l = -(x - \frac{\underset{as}{\overset{l}{x}}_j + \underset{as}{\overset{l+1}{x}}_j}{2})\sin\theta_l + (y - \frac{\underset{as}{\overset{l}{y}}_j + \underset{as}{\overset{l+1}{y}}_j}{2})\cos\theta_l \qquad l = 1,2,3,4 \tag{6.17}
$$

where θ_l is equal to the orientation of side l of the area-sink

$$
\theta_l = \arctan(\frac{\underset{as}{\overset{l+1}{y}}_j - \underset{as}{\overset{l}{y}}_j}{\underset{as}{\overset{l+1}{x}}_j - \underset{as}{\overset{l}{x}}_j}) \qquad l = 1,2,3,4 \quad -\pi < \theta_l \le \pi \tag{6.18}
$$

The range for the arctangent is from $-\pi$ to π.

The potential for area-sink j can be written as

$$
\underset{0}{\epsilon}_j \underset{as0}{\Lambda}_j(x,y,t) \tag{6.19}
$$

The potential for all $\underset{as}{m}$ area-sinks is equal to

$$\sum_{j=1}^{\underset{as}{m}}[\underset{0}{\epsilon_j}\underset{as0}{\Lambda_j}(x,y,t)] \tag{6.20}$$

line-segments with specified head.

Line-segments with a specified head are represented in the model by line-sinks with an a priori unknown strength. A number of line-sinks with different starting times is placed on one line-segment. The subcript $_{sh}$ is used to indicate line-segments with a specified head.

Let the head specified line-segment number j have end-points $(\underset{sh}{\overset{1}{x}_j}, \underset{sh}{\overset{1}{y}_j})$ and $(\underset{sh}{\overset{2}{x}_j}, \underset{sh}{\overset{2}{y}_j})$. Define a function $\underset{ls0}{\Theta_j}$ for a linesink on the boundary segment that has a starting time $\overset{k}{t}$ (compare (6.4))

$$\begin{aligned}
\underset{ls0}{\Theta_j}(x,y,t,\overset{k}{t}) =& 0 && t \le \overset{k}{t} \\
\underset{ls0}{\Theta_j}(x,y,t,\overset{k}{t}) =& -\frac{\sqrt{\alpha(t-\overset{k}{t})}}{2\sqrt{\pi}} e^{-\frac{\eta^2}{4\alpha(t-\overset{k}{t})}} \{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha(t-\overset{k}{t})}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha(t-\overset{k}{t})}})\} \\
&+\frac{1}{4\pi}\{(\xi-\overset{2}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha(t-\overset{k}{t})}) - (\xi-\overset{1}{\xi})\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha(t-\overset{k}{t})})\} \\
&+\frac{\eta}{2\sqrt{\pi}}\int_{\frac{\eta}{2\sqrt{\alpha(t-\overset{k}{t})}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\,du && t > \overset{k}{t}
\end{aligned} \tag{6.21}$$

where $\overset{1}{\xi}$, $\overset{2}{\xi}$, ξ and η are given below (compare (6.5) through (6.9))

$$\overset{1}{\xi} = -\frac{1}{2}\sqrt{(\underset{sh}{\overset{2}{x}_j}-\underset{sh}{\overset{1}{x}_j})^2+(\underset{sh}{\overset{2}{y}_j}-\underset{sh}{\overset{1}{y}_j})^2} \tag{6.22}$$

$$\overset{2}{\xi} = \frac{1}{2}\sqrt{(\underset{sh}{\overset{2}{x}_j}-\underset{sh}{\overset{1}{x}_j})^2+(\underset{sh}{\overset{2}{y}_j}-\underset{sh}{\overset{1}{y}_j})^2} \tag{6.23}$$

where $\sqrt{(\underset{sh}{\overset{2}{x}_j}-\underset{sh}{\overset{1}{x}_j})^2+(\underset{sh}{\overset{2}{y}_j}-\underset{sh}{\overset{1}{y}_j})^2}$ is the length of the element.

$$\xi = (x-\frac{\underset{sh}{\overset{1}{x}_j}+\underset{sh}{\overset{2}{x}_j}}{2})\cos\theta + (y-\frac{\underset{sh}{\overset{1}{y}_j}+\underset{sh}{\overset{2}{y}_j}}{2})\sin\theta \tag{6.24}$$

$$\eta = -(x-\frac{\underset{sh}{\overset{1}{x}_j}+\underset{sh}{\overset{2}{x}_j}}{2})\sin\theta + (y-\frac{\underset{sh}{\overset{1}{y}_j}+\underset{sh}{\overset{2}{y}_j}}{2})\cos\theta \tag{6.25}$$

where θ is equal to the orientation of the line-segment

$$\theta = \arctan(\frac{\underset{sh}{\overset{2}{y}_j}-\underset{sh}{\overset{1}{y}_j}}{\underset{sh}{\overset{2}{x}_j}-\underset{sh}{\overset{1}{x}_j}}) \qquad -\pi < \theta_l \le \pi \tag{6.26}$$

where the range for the arctangent is from $-\pi$ to π.

With (6.21) the potential for one line-sink with a priori unknown strength $\overset{k}{\underset{0}{\sigma}}_j$ can be written as

$$\overset{k}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(x, y, t, \overset{k-1}{t}) \tag{6.27}$$

The kth line-sink (with strength $\overset{k}{\underset{0}{\sigma}}_j$) at line-segment j is given the starting time $\overset{k-1}{t}$ for reasons that will become clear when the equations are set up later in this chapter. The potential for all n line-sinks at line-segment j is equal to

$$\sum_{k=1}^{n}[\overset{k}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(x, y, t, \overset{k-1}{t})] \tag{6.28}$$

The potential for all n line-sinks at all $\underset{sh}{m}$ line-segments with specified head is given by

$$\sum_{j=1}^{\underset{sh}{m}} \sum_{k=1}^{n}[\overset{k}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(x, y, t, \overset{k-1}{t})] \tag{6.29}$$

transient potential in model.

The elements included in the model are wells, line-sinks and area-sinks. It further contains line-segments with a specified head, far-field functions and the initial steady state. The potential can be written as

$$\Phi = \sum_{j=1}^{\underset{wl}{m}}[\underset{0}{Q}_j \underset{wl0}{\Lambda}_j(x, y, t)] + \sum_{j=1}^{\underset{ls}{m}}[\underset{0}{\sigma}_j \underset{ls0}{\Lambda}_j(x, y, t)] + \sum_{j=1}^{\underset{as}{m}}[\underset{0}{\epsilon}_j \underset{as0}{\Lambda}_j(x, y, t)]$$
$$+ \sum_{j=1}^{\underset{sh}{m}} \sum_{k=1}^{n}[\overset{k}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(x, y, t, \overset{k-1}{t})] + \underset{p}{\Phi} + \underset{r}{\Phi} + \overset{\text{steady}}{\underset{o}{\Phi}} \tag{6.30}$$

where (6.3), (6.11), (6.20) and (6.29) have been included. The far-field functions for changing the constant and the discharge at infinity are indicated by $\underset{p}{\Phi}$ (5.11) and $\underset{r}{\Phi}$ (5.19) respectively. The initial steady state is represented by $\overset{\text{steady}}{\underset{o}{\Phi}}$.

determining unknown strengths

There are several steps in obtaining a complete solution for a transient groundwater flow problem:

- determining the initial steady state
- choosing the parameters of the functions to adjust the far-field from the initial steady state to the final steady state
- solving for the transient part of the potential

The initial steady state is modeled with the analytic element method for steady flow (see chapter 3 and Strack, 1988). The flow for the steady state is simulated before the transient modeling is started, so that the potential $\overset{\text{steady}}{\underset{o}{\Phi}}$ for the initial steady state is known as a function of position.

The functions to adjust the far-field are only useful in the case that the model is applied to the transition from the initial steady state to a new steady state. In that case it is not necessary to use them, unless the modeled area is not extended far enough beyond the area of interest (see chapter 5). If the final steady state is modeled, using the analytic element method for steady flow again, it can be decided whether the far-field functions need to be included. If so, the parameters can be determined from this final steady state, as is described in chapter 5.

equations.

The potential for the model is given by (6.30). The potential contains n line-sinks at each of the $\underset{sh}{m}$ line-segments. The strengths of these line-sinks are not known beforehand. This amounts to n times $\underset{sh}{m}$ a priori unknown strengths. An equal number of equations is needed to determine these strengths. The equations are obtained from the specified heads. There are $\underset{sh}{m}$ line-segments at which the heads are specified. A number of n different times $\overset{s}{t}$ is selected, at which the conditions for the specified heads will be set. The times $\overset{s}{t}$ are ordered such that

$$\overset{1}{t} < \overset{2}{t} < .... < \overset{n-1}{t} < \overset{n}{t} \tag{6.31}$$

moreover the first time is greater than the time $\overset{0}{t}$ at which the transient modeling starts

$$\overset{0}{t} < \overset{1}{t} < \overset{2}{t} < .... < \overset{n-1}{t} < \overset{n}{t} \tag{6.32}$$

The times $\overset{s}{t}$ with $0 \leq s \leq n-1$ are chosen to coincide with the starting times of the line-sinks at the line-segments (see (6.27)).

Let the value of the head specified at line-segment l be equal to $\underset{sh}{\varphi_l}$. This value of the head can be translated into the value $\underset{sh}{\Phi_l}$ for the potential, by means of either (2.10) or (2.14), depending on the type of flow. The condition is set that the potential has the value $\underset{sh}{\Phi_l}$ at the center $(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l})$ of line-segment number l. The $\underset{sh}{m}$ known potentials at n different times give n times $\underset{sh}{m}$ equations that are needed to determine the potential for the model at these n times

$$\Phi(\underset{sh}{\overset{c}{x}_l}, \underset{s}{\overset{c}{y}_l}, \overset{s}{t}) = \underset{sh}{\Phi_l} \qquad l = 1, 2, ...\underset{sh}{m} \quad s = 1, 2, ...n \tag{6.33}$$

Using (6.30), the conditions can be written as

$$\begin{aligned}
&\sum_{j=1}^{\underset{wl}{m}}[\underset{0}{Q_j}\underset{wl0}{\Lambda_j}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t})] + \sum_{j=1}^{\underset{ls}{m}}[\underset{0}{\sigma_j}\underset{ls0}{\Lambda_j}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t})] + \sum_{j=1}^{\underset{as}{m}}[\underset{0}{\epsilon_j}\underset{as0}{\Lambda_j}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t})] \\
&\quad + \sum_{j=1}^{\underset{sh}{m}}\sum_{k=1}^{n}[\underset{0}{\overset{k}{\sigma}_j}\underset{ls0}{\Theta_j}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t}, \overset{k-1}{t})] \\
&\quad + \underset{p}{\Phi} + \underset{r}{\Phi}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t}) + \underset{o}{\overset{\text{steady}}{\Phi}}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}) \\
&= \underset{sh}{\Phi_l} \qquad\qquad l = 1, 2, ...\underset{sh}{m} \quad s = 1, 2, ...n
\end{aligned} \tag{6.34}$$

or

$$\begin{aligned}
&\sum_{j=1}^{\underset{sh}{m}}\sum_{k=1}^{n}[\underset{0}{\overset{k}{\sigma}_j}\underset{ls0}{\Theta_j}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t}, \overset{k-1}{t})] \\
&= \underset{sh}{\Phi_l} - \sum_{j=1}^{\underset{wl}{m}}[\underset{0}{Q_j}\underset{wl0}{\Lambda_j}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t})] - \sum_{j=1}^{\underset{ls}{m}}[\underset{0}{\sigma_j}\underset{ls0}{\Lambda_j}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t})] - \sum_{j=1}^{\underset{as}{m}}[\underset{0}{\epsilon_j}\underset{as0}{\Lambda_j}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t})] \\
&\quad - \underset{p}{\Phi}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t}) - \underset{r}{\Phi}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}, \overset{s}{t}) - \underset{o}{\overset{\text{steady}}{\Phi}}(\underset{sh}{\overset{c}{x}_l}, \underset{sh}{\overset{c}{y}_l}) \qquad l = 1, 2, ...\underset{sh}{m} \quad s = 1, 2, ...n
\end{aligned} \tag{6.35}$$

where the right hand side of the equation is known. The n times $\underset{sh}{m}$ equations (6.35) do not have to be solved simultaneously. From (6.21) it can be seen that

$$\underset{ls0}{\Theta}_j(x,y,\overset{s}{t},\overset{k-1}{t}) = 0 \qquad \overset{s}{t} \leq \overset{k-1}{t} \tag{6.36}$$

and with (6.32)

$$\underset{ls0}{\Theta}_j(x,y,\overset{s}{t},\overset{k-1}{t}) = 0 \qquad s < k \tag{6.37}$$

Thus the potential for line-sinks number k at a line-segment is equal to zero at the times $\overset{s}{t}$ if k is greater than s, no matter what the value of the strength of the line-sink is. Therefore the equations (6.35) can be written as follows for $s = 1$

$$\begin{aligned}
&\sum_{j=1}^{\underset{sh}{m}} [\underset{0}{\overset{1}{\sigma}}_j \underset{ls0}{\Theta}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t}, \overset{0}{t})] \\
&= \underset{sh}{\Phi}_l - \sum_{j=1}^{\underset{ul}{m}} [\underset{0}{Q}_j \underset{ul0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t})] - \sum_{j=1}^{\underset{ls}{m}} [\underset{0}{\sigma}_j \underset{ls0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t})] - \sum_{j=1}^{\underset{as}{m}} [\underset{0}{\epsilon}_j \underset{as0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t})] \\
&\quad - \underset{p}{\Phi}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t}) - \underset{r}{\Phi}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t}) - \underset{o}{\overset{\text{steady}}{\Phi}}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l) \qquad l = 1,2,....\underset{sh}{m}
\end{aligned} \tag{6.38}$$

This represents $\underset{sh}{m}$ equations with $\underset{sh}{m}$ unknowns. This system of equations can be written in matrix form

$$\begin{pmatrix}
A(1,1) & A(1,2) & \dots & A(1,j) & \dots & A(1,\underset{sh}{m}) \\
A(2,1) & A(2,2) & \dots & A(2,j) & \dots & A(2,\underset{sh}{m}) \\
\vdots & \vdots & \ddots & \vdots & & \vdots \\
A(l,1) & A(l,2) & \dots & A(l,j) & \dots & A(l,\underset{sh}{m}) \\
\vdots & \vdots & & \vdots & \ddots & \vdots \\
A(\underset{sh}{m},1) & A(\underset{sh}{m},2) & \dots & A(\underset{sh}{m},j) & \dots & A(\underset{sh}{m},\underset{sh}{m})
\end{pmatrix}
\begin{pmatrix} X(1) \\ X(2) \\ \vdots \\ X(j) \\ \vdots \\ X(\underset{sh}{m}) \end{pmatrix}
=
\begin{pmatrix} B(1) \\ B(2) \\ \vdots \\ B(l) \\ \vdots \\ B(\underset{sh}{m}) \end{pmatrix} \tag{6.39}$$

where

$$A(l,j) = \underset{ls0}{\Theta}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t}, \overset{0}{t}) \qquad l = 1,2,....\underset{sh}{m} \quad j = 1,2,....\underset{sh}{m} \tag{6.40}$$

$$X(j) = \underset{0}{\overset{1}{\sigma}}_j \qquad j = 1,2,....\underset{sh}{m} \tag{6.41}$$

$$\begin{aligned}
B(l) =& \underset{sh}{\Phi}_l - \sum_{j=1}^{\underset{ul}{m}} [\underset{0}{Q}_j \underset{ul0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t})] - \sum_{j=1}^{\underset{ls}{m}} [\underset{0}{\sigma}_j \underset{ls0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t})] - \sum_{j=1}^{\underset{as}{m}} [\underset{0}{\epsilon}_j \underset{as0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t})] \\
&- \underset{p}{\Phi}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t}) - \underset{r}{\Phi}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{1}{t}) - \underset{o}{\overset{\text{steady}}{\Phi}}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l) \qquad l = 1,2,....\underset{sh}{m}
\end{aligned} \tag{6.42}$$

The $\underset{sh}{m}$ strengths $\underset{0}{\overset{1}{\sigma}}_j$ can be determined from the matrix equation (6.39) using Gaussian elimination (see Wylie and Barrett (1982) for instance).

With this solution the potential now is known for times $t \leq \overset{2}{t}$. The time $\overset{2}{t}$ is the time at which the next elements start with strengths that are yet undetermined. The potential is known as a funcion of position and time up to $\overset{s}{t}$ after s solutions.

The equations (6.35) give for $s = 2$

$$\begin{aligned}
&\sum_{j=1}^{\underset{sh}{m}}[\overset{2}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{2}{t},\overset{1}{t})] \\
&= \underset{sh}{\Phi}_l - \sum_{j=1}^{\underset{wl}{m}}[\underset{0}{Q}_j \underset{wl0}{\Lambda}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{2}{t})] - \sum_{j=1}^{\underset{ls}{m}}[\underset{0}{\sigma}_j \underset{ls0}{\Lambda}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{2}{t})] - \sum_{j=1}^{\underset{as}{m}}[\underset{0}{\epsilon}_j \underset{as0}{\Lambda}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{2}{t})] \\
&\quad - \underset{p}{\Phi}(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{2}{t}) - \underset{r}{\Phi}(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{2}{t}) - \overset{\text{steady}}{\underset{o}{\Phi}}(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l) \\
&\quad - \sum_{j=1}^{\underset{sh}{m}}[\overset{1}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{2}{t},\overset{0}{t})] \qquad l = 1,2,....\underset{sh}{m}
\end{aligned} \tag{6.43}$$

This is again a system of $\underset{sh}{m}$ equations. In this system only the $\underset{sh}{m}$ strengths $\overset{2}{\underset{0}{\sigma}}_j$ are unknown, since the strengths $\overset{1}{\underset{0}{\sigma}}_j$ are known from (6.38). Next the strengths $\overset{3}{\underset{0}{\sigma}}_j$ can be solved from the equations that result from (6.35) for $s = 3$

$$\begin{aligned}
&\sum_{j=1}^{\underset{sh}{m}}[\overset{3}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{3}{t},\overset{2}{t})] \\
&= \underset{sh}{\Phi}_l - \sum_{j=1}^{\underset{wl}{m}}[\underset{0}{Q}_j \underset{wl0}{\Lambda}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{3}{t})] - \sum_{j=1}^{\underset{ls}{m}}[\underset{0}{\sigma}_j \underset{ls0}{\Lambda}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{3}{t})] - \sum_{j=1}^{\underset{as}{m}}[\underset{0}{\epsilon}_j \underset{as0}{\Lambda}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{3}{t})] \\
&\quad - \underset{p}{\Phi}(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{3}{t}) - \underset{r}{\Phi}(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{3}{t}) - \overset{\text{steady}}{\underset{o}{\Phi}}(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l) \\
&\quad - \sum_{j=1}^{\underset{sh}{m}}\sum_{k=1}^{2}[\overset{k}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{3}{t},\overset{k-1}{t})] \qquad l = 1,2,....\underset{sh}{m}
\end{aligned} \tag{6.44}$$

So that in general the strengths $\overset{s}{\underset{0}{\sigma}}_j$ can be solved from the following system of equations, once the strengths $\overset{k}{\underset{0}{\sigma}}_j$ have been determined up to and including $k = s - 1$

$$\begin{aligned}
&\sum_{j=1}^{\underset{sh}{m}}[\overset{s}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{s}{t},\overset{s-1}{t})] \\
&= \underset{sh}{\Phi}_l - \sum_{j=1}^{\underset{wl}{m}}[\underset{0}{Q}_j \underset{wl0}{\Lambda}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{s}{t})] - \sum_{j=1}^{\underset{ls}{m}}[\underset{0}{\sigma}_j \underset{ls0}{\Lambda}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{s}{t})] - \sum_{j=1}^{\underset{as}{m}}[\underset{0}{\epsilon}_j \underset{as0}{\Lambda}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{s}{t})] \\
&\quad - \underset{p}{\Phi}(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{s}{t}) - \underset{r}{\Phi}(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{s}{t}) - \overset{\text{steady}}{\underset{o}{\Phi}}(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l) \\
&\quad - \sum_{j=1}^{\underset{sh}{m}}\sum_{k=1}^{s-1}[\overset{k}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(\overset{c}{\underset{sh}{x}}_l,\overset{c}{\underset{sh}{y}}_l,\overset{s}{t},\overset{k-1}{t})] \qquad l = 1,2,....\underset{sh}{m} \quad s = 1,2,....n
\end{aligned} \tag{6.45}$$

And once these strengths have been obtained, the potential is known as a function of position and time for $\overset{0}{t} \leq t \leq \overset{s}{t}$.

The elements of the matrices A, X and B in equation (6.39) can be written in general form applicable to any time $\overset{s}{t}$. The following three equations replace (6.40), (6.41) and (6.42) respectively, which apply to time $\overset{1}{t}$ only.

$$A(l,j) = \underset{ls0}{\Theta}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t}, \overset{s-1}{t}) \qquad l = 1,2,....\underset{sh}{m} \quad j = 1,2,....\underset{sh}{m} \tag{6.46}$$

$$X(j) = \overset{s}{\underset{0}{\sigma}}_j \qquad j = 1,2,....\underset{sh}{m} \quad s = 1,2,....\underset{sh}{m} \tag{6.47}$$

$$\begin{aligned} B(l) =& \underset{sh}{\Phi}_l - \sum_{j=1}^{\underset{ul}{m}}[\underset{0}{Q}_j \underset{ul0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t})] - \sum_{j=1}^{\underset{ls}{m}}[\underset{0}{\sigma}_j \underset{ls0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t})] - \sum_{j=1}^{\underset{as}{m}}[\underset{0}{\epsilon}_j \underset{as0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t})] \\ &- \underset{p}{\Phi}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t}) - \underset{r}{\Phi}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t}) - \overset{\text{steady}}{\underset{o}{\Phi}}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l) \\ &- \sum_{j=1}^{\underset{sh}{m}} \sum_{k=1}^{s-1} [\overset{k}{\underset{0}{\sigma}}_j \underset{ls0}{\Theta}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t}, \overset{k-1}{t})] \qquad l = 1,2,....\underset{sh}{m} \quad s = 1,2,....n \end{aligned} \tag{6.48}$$

Now it can be seen why the kth line-sink at a line-segment was given the starting time $\overset{k-1}{t}$ in equation (6.27) and why the solving times were chosen to coincide with the starting times. These choices made it possible to solve n times a system of $\underset{sh}{m}$ equations in stead of one time the original system (6.35) of n times $\underset{sh}{m}$ equations, which means an large reduction in memory and computation time in a computer program.

Moreover, now that the solution is obtained gradually stepping forward in time, it is possible to adjust the size of the next time-step, based on the last results. One can also monitor the potential solution and decide earlier if some of the elements in the model need to be changed.

Above it has been assumed that all line-segments are active during the entire period that is modeled. This is not necessary. Line-segments can be added or removed without changing the above procedure significantly. The difference is that $\underset{sh}{m}$ is not the same for every system of equations (6.45) so that the matrices (6.46), (6.47) and (6.48) vary in size.

However it is advantageous to have all line-segments active during the entire proces if a constant step-size is used. The elements (6.46) of the matrix A are the same for every solution, if the time-step $\overset{s+1}{t} - \overset{s}{t}$ is constant since

$$\underset{ls0}{\Theta}_j(x, y, \overset{k}{t}, \overset{k-1}{t}) = \underset{ls0}{\Theta}_j(x, y, \overset{k}{t} - \overset{k-1}{t}, 0) \tag{6.49}$$

as can be seen from (6.21).

So with a constant time step another large reduction in calculation time can be achieved. The Gaussian form of the matrix A is obtained when the first solution is determined. The column matrix B is determined in the next time-steps and the solutions can be obtained using the same Gaussian form of the matrix.

It has also been assumed that the heads specified at the line-segments are constant in time. Varying the specified head is easily incorporated. The head is equal to $\overset{k}{\underset{sh}{\varphi}}_j$ at time $\overset{k}{t}$, in stead of equal to $\underset{sh}{\varphi}_j$ at all times of solving. The corresponding potential $\overset{k}{\underset{sh}{\Phi}}_j$ is obtained with (2.11) or

(2.14) depending on the type of flow. This potential replaces the potential $\underset{sh}{\Phi}_j$ in the system of equations (6.45), which corresponds to a change of B in the matrix equation (6.38). In stead of (6.48) it is equal to

$$B(l) = \underset{l}{\overset{sh}{\Phi}}_j - \sum_{j=1}^{\underset{wl}{m}} [\underset{0}{Q_j} \underset{wl0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t})] - \sum_{j=1}^{\underset{ls}{m}} [\underset{0}{\sigma_j} \underset{ls0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t})] - \sum_{j=1}^{\underset{as}{m}} [\underset{0}{\epsilon_j} \underset{as0}{\Lambda}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t})]$$
$$- \underset{p}{\Phi}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t}) - \underset{r}{\Phi}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t}) - \underset{o}{\overset{\text{steady}}{\Phi}}(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l)$$
$$- \sum_{j=1}^{\underset{sh}{m}} \sum_{k=1}^{s-1} [\underset{0}{\overset{k}{\sigma}}_j \underset{ls0}{\Theta}_j(\underset{sh}{\overset{c}{x}}_l, \underset{sh}{\overset{c}{y}}_l, \overset{s}{t}, \overset{k-1}{t})] \qquad l = 1, 2, \ldots . \underset{sh}{m} \quad s = 1, 2, \ldots . n \tag{6.50}$$

example.

The process of solving will be illustrated below for a simple example (see figure 6.1), which has been used in chapter 5 also. The potential is constant in the initial steady state. At time zero a well starts to pump a constant amount. At a line-segment the head is specified to remain at the initial level.

The far-field functions will not be considered here. It has been described in chapter 5 how these are used. The description was illustrated with examples involving the same simple case of flow.

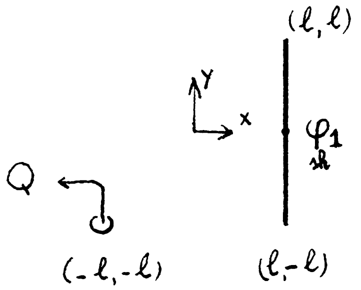

figure 6.1. *Problem for example of solving*

The potential for the initial steady state is equal to

$$\underset{o}{\overset{\text{steady}}{\Phi}} = \underset{co}{\Phi} \tag{6.51}$$

where the subscript $_o$ indicates the initial steady state and $_c$ the constant. The potential at the center (x_c, y_c) of the line-segment remains equal to this value

$$\underset{sh}{\Phi_1} = \underset{co}{\Phi} \tag{6.52}$$

The discharge of the well is Q_w and starts at $t = 0$. The first line-sink at the line-segment starts at this time also. The potential for this case of flow is given by (compare (6.30))

$$\Phi(x, y, t) = Q_w \underset{wl0}{\Lambda}_1(x, y, t) + \sum_{k=1}^{n} \underset{0}{\overset{k}{\sigma}}_1 \underset{ls0}{\Theta}_1(x, y, t, \overset{k-1}{t}) + \underset{co}{\Phi} \tag{6.53}$$

The potential up to the first time of solving is

$$\Phi(x, y, t) = Q_w \underset{wl0}{\Lambda}_1(x, y, t) + \underset{0}{\overset{1}{\sigma}}_1 \underset{ls0}{\Theta}_1(x, y, t, 0) + \underset{co}{\Phi} \qquad 0 \le t \le \overset{1}{t} \tag{6.54}$$

where $\overset{1}{\underset{0}{\sigma}}_1$ has to be determined from equation (6.33)

$$\Phi(x_c, y_c, \overset{1}{t}, 0) = \underset{sh}{\Phi}_1 \tag{6.55}$$

Using (6.52) (compare (6.38)), this can be written as,

$$\overset{1}{\underset{0}{\sigma}}_1 \underset{ls0}{\Theta}_1(x_c, y_c, \overset{1}{t}, 0) = \underset{sh}{\Phi}_1 - Q_w \underset{ul0}{\Lambda}_1(x_c, y_c, \overset{1}{t}) - \underset{co}{\Phi} \tag{6.56}$$

so that

$$\overset{1}{\underset{0}{\sigma}}_1 = \frac{\underset{sh}{\Phi}_1 - Q_w \underset{ul0}{\Lambda}_1(x_c, y_c, \overset{1}{t}) - \underset{co}{\Phi}}{\underset{ls0}{\Theta}_1(x_c, y_c, \overset{1}{t}, 0)} \tag{6.57}$$

The strength $\overset{2}{\underset{0}{\sigma}}_1$ can be determined from the equation (6.43)

$$\overset{2}{\underset{0}{\sigma}}_1 \underset{ls0}{\Theta}_1(x_c, y_c, \overset{2}{t}, \overset{1}{t}) = \underset{sh}{\Phi}_1 - Q_w \underset{ul0}{\Lambda}_1(x_c, y_c, \overset{2}{t}) - \underset{co}{\Phi} - \overset{1}{\underset{0}{\sigma}}_1 \underset{ls0}{\Theta}_1(x_c, y_c, \overset{2}{t}, 0) \tag{6.58}$$

which gives

$$\overset{2}{\underset{0}{\sigma}}_1 = \frac{\underset{sh}{\Phi}_1 - Q_w \underset{ul0}{\Lambda}_1(x_c, y_c, \overset{2}{t}) - \underset{co}{\Phi} - \overset{1}{\underset{0}{\sigma}}_1 \underset{ls0}{\Theta}_1(x_c, y_c, \overset{2}{t}, 0)}{\underset{ls0}{\Theta}_1(x_c, y_c, \overset{2}{t}, 0)} \tag{6.59}$$

so that the potential (6.53) is known now up to time $\overset{3}{t}$. To extend it up to $\overset{4}{t}$ equation (6.44) is used

$$\begin{aligned} &\overset{3}{\underset{0}{\sigma}}_1 \underset{ls0}{\Theta}_1(x_c, y_c, \overset{3}{t}, \overset{2}{t}) \\ &\quad = \underset{sh}{\Phi}_1 - Q_w \underset{ul0}{\Lambda}_1(x_c, y_c, \overset{3}{t}) - \underset{co}{\Phi} - \overset{1}{\underset{0}{\sigma}}_1 \underset{ls0}{\Theta}_1(x_c, y_c, \overset{3}{t}, 0) - \overset{2}{\underset{0}{\sigma}}_1 \underset{ls0}{\Theta}_1(x_c, y_c, \overset{3}{t}, \overset{1}{t}) \end{aligned} \tag{6.60}$$

from which the strength of the third line-sink is obtained

$$\overset{3}{\underset{0}{\sigma}}_1 = \frac{\underset{sh}{\Phi}_1 - Q_w \underset{ul0}{\Lambda}_1(x_c, y_c, \overset{3}{t}) - \underset{co}{\Phi} - \sum_{k=1}^{2} \overset{k}{\underset{0}{\sigma}}_1 \underset{ls0}{\Theta}_1(x_c, y_c, \overset{3}{t}, \overset{k-1}{t})}{\underset{ls0}{\Theta}_1(x_c, y_c, \overset{3}{t}, \overset{2}{t})} \tag{6.61}$$

and the process can be continued using (6.45) to give the next strengths up to n

$$\overset{s}{\underset{0}{\sigma}}_1 = \frac{\underset{sh}{\Phi}_1 - Q_w \underset{ul0}{\Lambda}_1(x_c, y_c, \overset{s}{t}) - \underset{co}{\Phi} - \sum_{k=1}^{s-1} \overset{k}{\underset{0}{\sigma}}_1 \underset{ls0}{\Theta}_1(x_c, y_c, \overset{s}{t}, \overset{k-1}{t})}{\underset{ls0}{\Theta}_1(x_c, y_c, \overset{s}{t}, \overset{s-1}{t})} \tag{6.62}$$

The resulting discharge at the line-segment is shown in figure 6.2. Contour plots of the potential at different times are shown in figure 6.3.

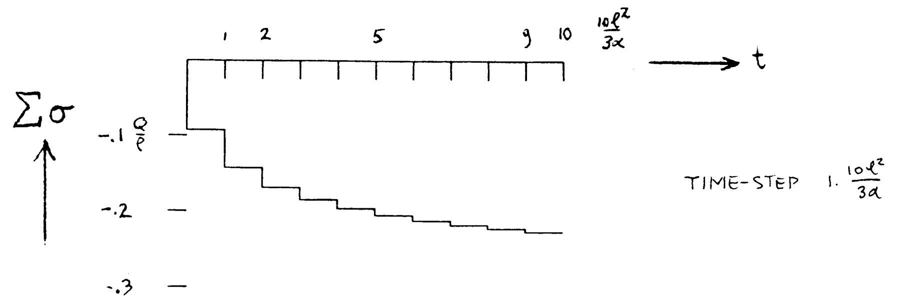

figure 6.2. *Summed discharge at line-segment*

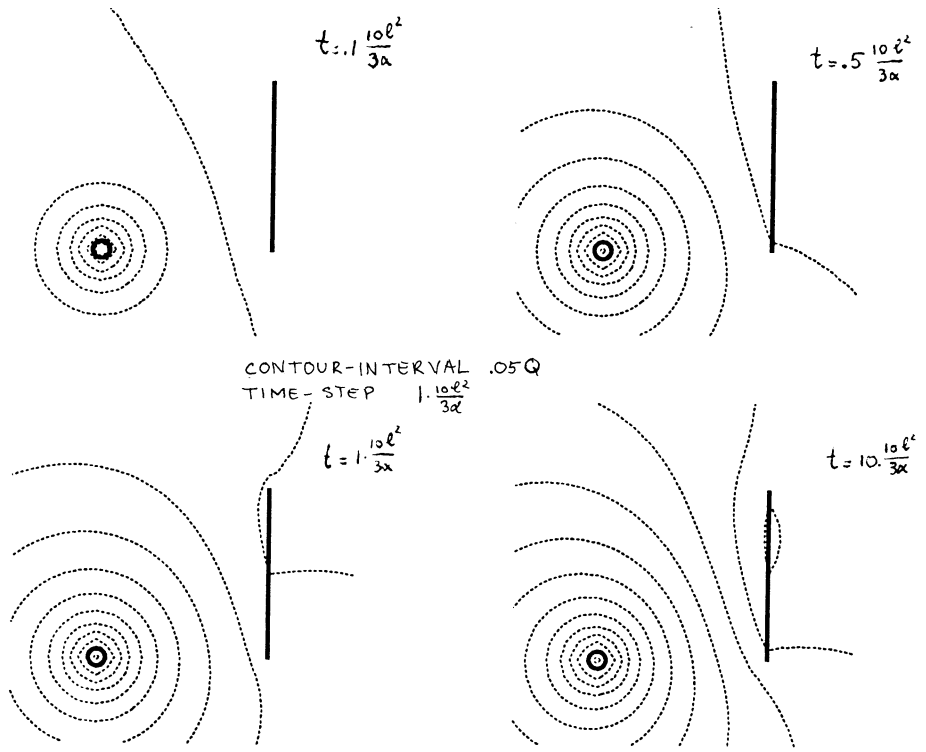

figure 6.3. *Equipotentials at different times*

time-steps and errors

The above results were given for one arbitrary size of the time-step. The influence of the time-step on the solution is considered in the following.

First the same case (see figure 6.1) will be examined. Solutions obtained with different step-sizes that are kept constant in the process of solving are compared. The effect of changing the size of the time-step is also shown.

Next the case will be considered that the well is not pumping and the head at the line-segment is suddenly raised. The discharge of the line-segment as a function of time is compared to that of the previous case. The cause of the transient effects at the line-segment has a quite different character. The well is located at some distance from the line-segment and its effect increases in time, while the sudden raising of the head occurs right at the line-segment and its effect decreases in time.

well and line-segment.

constant size of time step. The case of a well and a line-segment (see figure 6.1) was solved with a constant step size in the example above. The total distributed discharge and contour plots were given in figures 6.2 and 6.3 respectively. The potential at the center of the line-segment is plotted versus time in figure 6.4. The value of the specified potential coincides with the horizontal axis. The actual potential assumes this value only at the times of solving. During each time step the curve for the error has approximately the same shape. It increases sharply at the beginning of the interval, reaches a peak at about a fifth of the interval, and then decreases more gradually toward the end of the interval. The magnitude decreases with time.

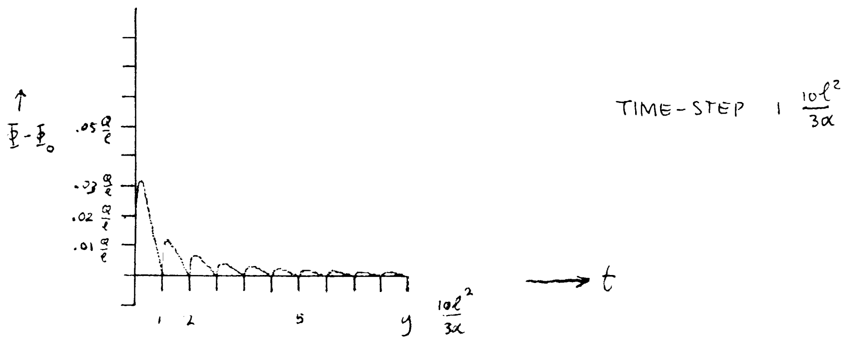

figure 6.4. *Potential at center of line-segment*

The problem was solved again using three different values for the time step, equal to .02, .2 and 2 times the step used above. In figure 6.5 the total distributed discharge at the line-segment is plotted versus time for the three different time steps. The three stair-case curves follow the same path. This indicates that the errors do not accumulate in time. In figure 6.6 the potentials at the center of the line-segment are plotted for solutions obtained with stepsizes with factors .5, 1 and 2 in stead of .02, .2 and 2, so that the errors are visible in the graph.

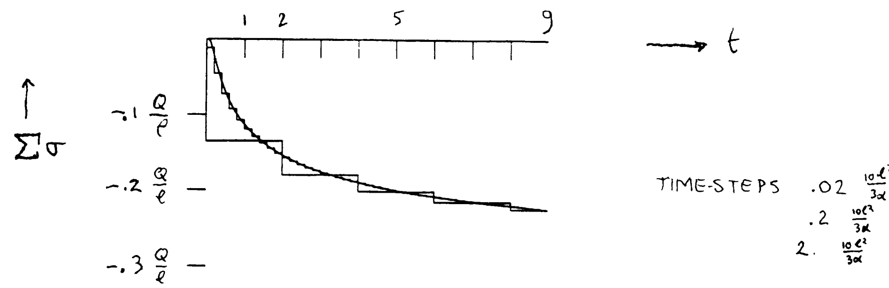

figure 6.5. *Distributed discharges at the line-segment for three different constant time steps*

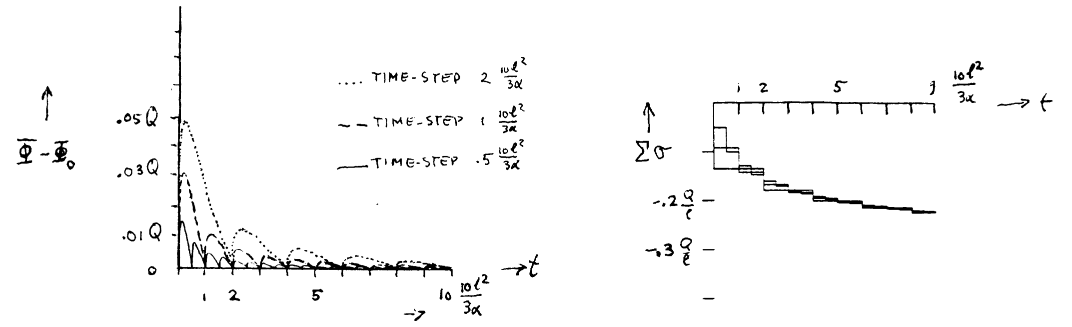

figure 6.6. *Potentials at the center of the line-segment for three different constant time steps*

increasing size of time step. The case of a well and a line-segment (see figure 6.1) is solved next with a time step that increases in time. The steps will be increased in such a way that the maximum error in a time step at the center of the line-segment is constant. In figure 6.7 the potentials at the center of the line-segment is given as a function of time for two different initial step sizes (compare figure 6.6). In figure 6.8 the corresponding distributed discharges of the line-segment are given (compare figure 6.5).

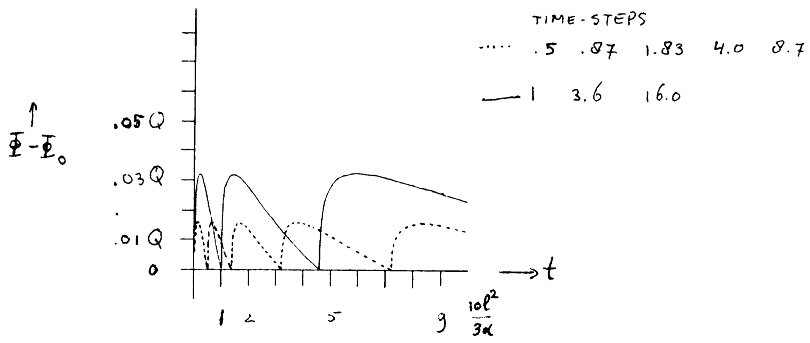

figure 6.7. *Potentials at the center of the line-segment for different increasing time steps*

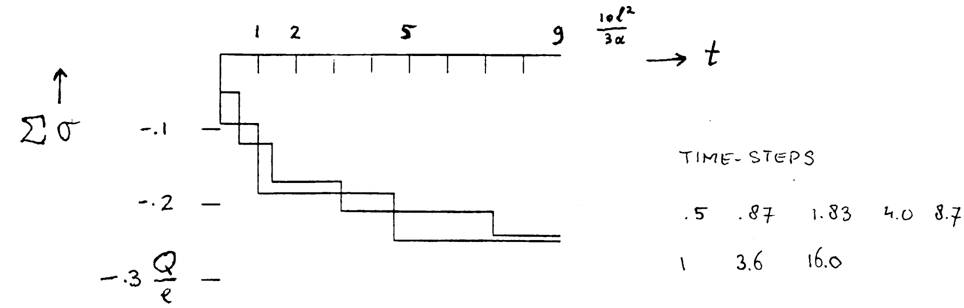

figure 6.8. *Distributed discharges at the line-segment for different increasing time steps*

varying size of the time step. In figure 6.9 the potentials at the center are given for different solutions with varying time-steps. The final time step is the same for the three solutions. The errors in this interval do not vary much between the three solutions. The difference of a factor four in the stepsizes translates into a factor of four thirds in the maximum error. The relatively small of influence of previous steps can also be seen from the the discharges at the line-segment, which are plotted versus the time in figure 6.10.

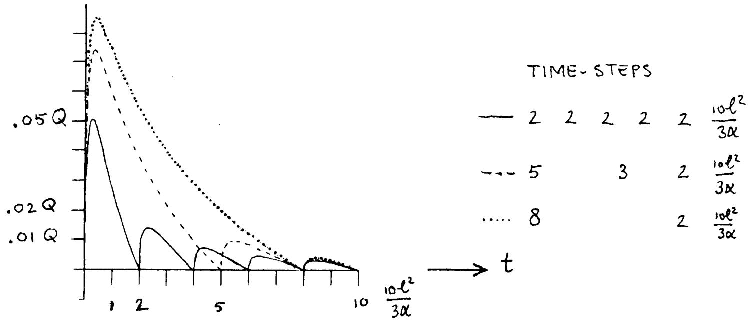

figure 6.9. *Potentials at the center of the line-segment for varying time steps*

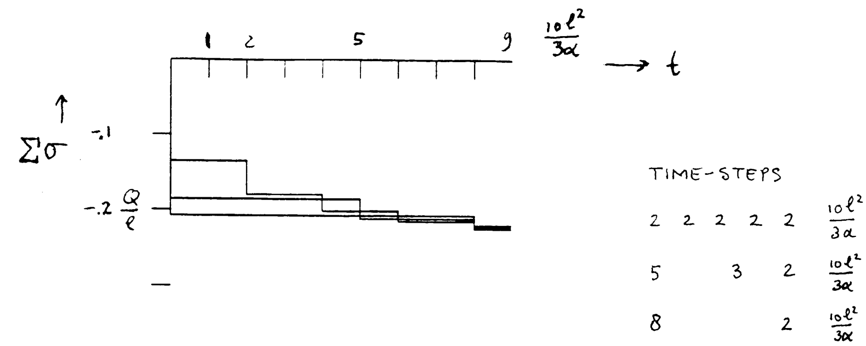

figure 6.10. *Distributed discharges at the line-segment for three time steps*

line-segment with suddenly increased head.

The case of a line-segment with a head that suddenly increases is discussed next. Initially the head is constant in space. At time zero the head at the line-segment is raised instantaneously. The amount of change is chosen such that the effect is comparable to the effect of the well on the line-segment. The new potential at the line-segment is indicated by Φ_1. The graphs for the potentials and the distributed discharges are given at the same scale as for the above case and the similar time-steps are used.

constant size of time step. In figure 6.11 the potential at the center of the line-segment is plotted versus the time. The initial level of the potential coincides with the horizontal axis. The specified potential at the line-segment is indicated by a horizontal line. The potential is equal to the specified value at the times of solving. It is lower in between these times; lower means in this case closer to the initial value. The corresponding distributed discharges at the line-segment are given in figure 6.12.

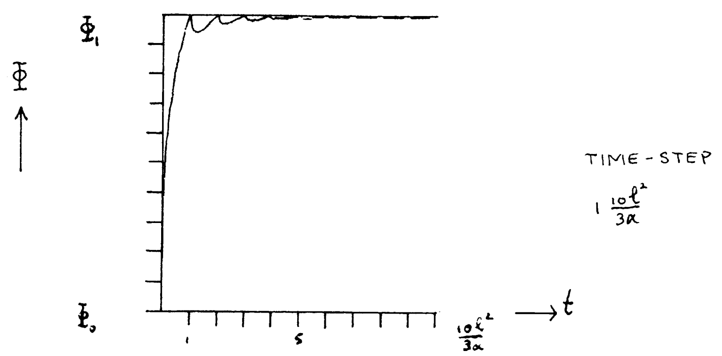

figure 6.11. *Potential at center of line-segment*

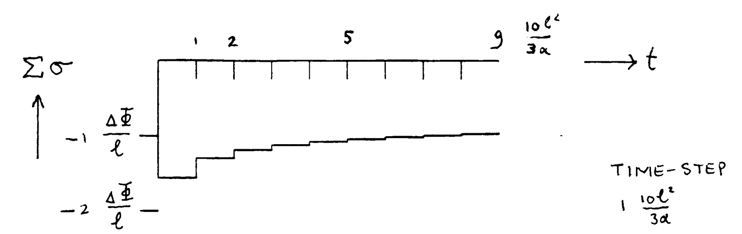

figure 6.12. *Distributed discharge at center of line segment*

The problem was solved with time steps that were .02, .2 and 2 times the size of the time step that was used for figure 6.11 and figure 6.12. The distributed discharges are given in figure 6.13. A different choice of time-steps was made for the plot of the potential at the center of the line-segment (see figure 6.14). The steps are now .5, 1 and 2 times the step in figure 6.11 so that the deviations from the specified value of the potential are visible in the graph.

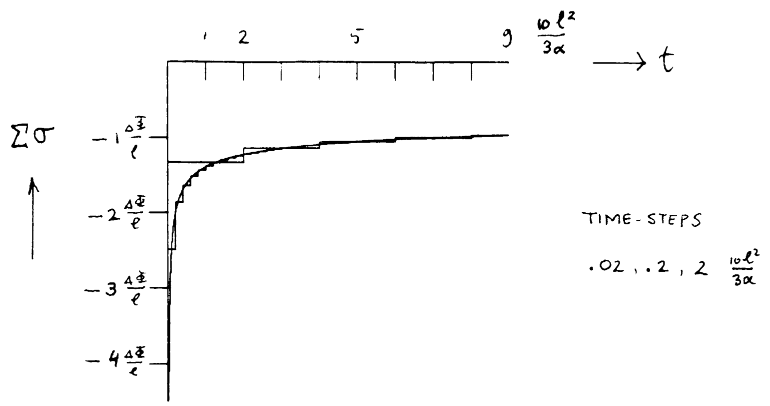

figure 6.13. *Distributed discharges at the line-segment for three different constant time steps*

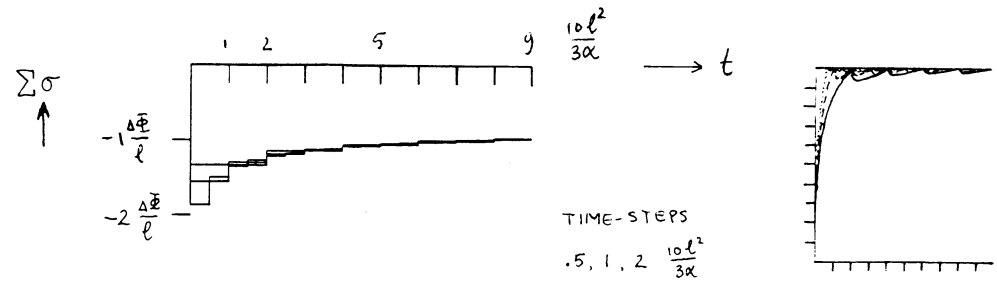

figure 6.14. *Potentials at the center of the line-segment for three different constant time steps*

increasing size of time step. The case of a line-segment with a suddenly raised head at the center is solved with an increasing step size. The steps are increased such way that the maximum error during the third and following time steps is the same as the maximum error in the second step, see figure 6.16. The corresponding distributed discharges are shown in figure 6.17.

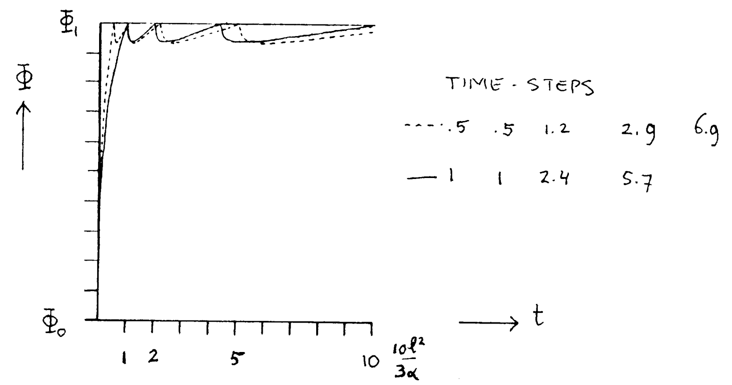

figure 6.15. *Potentials at the center of the line-segment for two different increasing time steps*

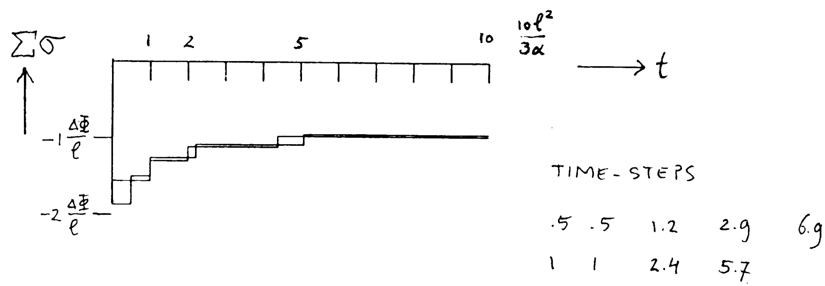

figure 6.16. *Distributed discharges at the line-segment for two different increasing time steps*

varying size of the time step. The case with a line-segment with suddenly increased head is solved with the same varying time steps as used for the case with a well and a line-segment shown in figure 6.9 and figure 6.10. The size of the final time step is equal for the three cases. The results are given in figure 6.18 and figure 6.19. Again the errors in the final time step are not very sensitive to the size of the previous time step.

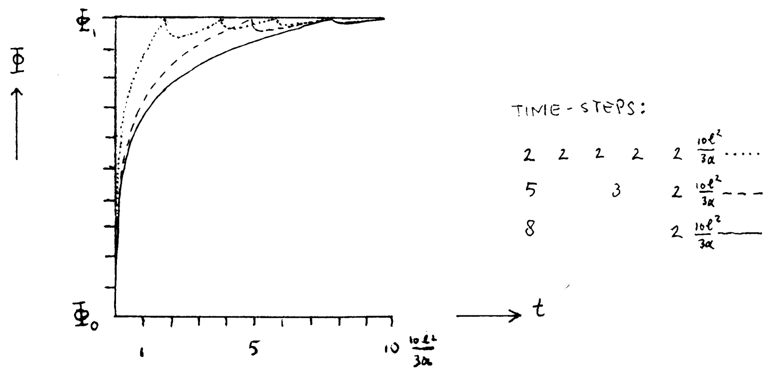

figure 6.18. *Potentials at the center of the line-segment for varying time steps*

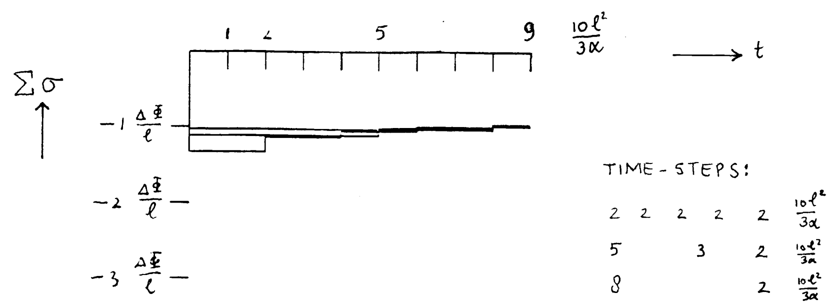

figure 6.19. *Distributed discharges at the line-segment for varying time steps*

discussion.

The cases of flow that were examined above are very simple. However, they contain elementary sources of transient flow, and a general conclusion can be drawn.

The step size can be increased gradually after the driving force for the transient effect has started to keep the errors at a constant level during the entire modeled period. The errors at larger times are not sensitive to the step sizes used at smaller times in the solution.

reducing computational effort

Elements are added to the potential at each time step. In this way the number of functions that have to be evaluated to get the value of the potential or the discharge at a point increases steadily with time. Measures have to be taken to prevent the computational times from becoming excessive.

One way to reduce the computational effort is to replace a number of line-sinks at one line-segment. Above it was seen that the errors correlated to the size of the time step mainly depend on the last time steps and hardly on the earlier ones. So for evaluating the potential at a certain time, the solution before that time does not need the detail it was solved with, but can be represented more coarsely. The influence of a group of elements at one location can be approximated by just two elements. The two elements start at the starting times of the first and last elements of the group. The total amount of water that is removed from the aquifer is the same and the value of the sum of the strengths at the starting time of the last element is the same (see figure 6.20). This is done by giving the first substitute well a strength equal to the weighted average of the discharges in the period of substitution. The discharge of the second substitute well is equal to the sum of the discharges of the wells that are being replaced minus the strength of the first substitute.

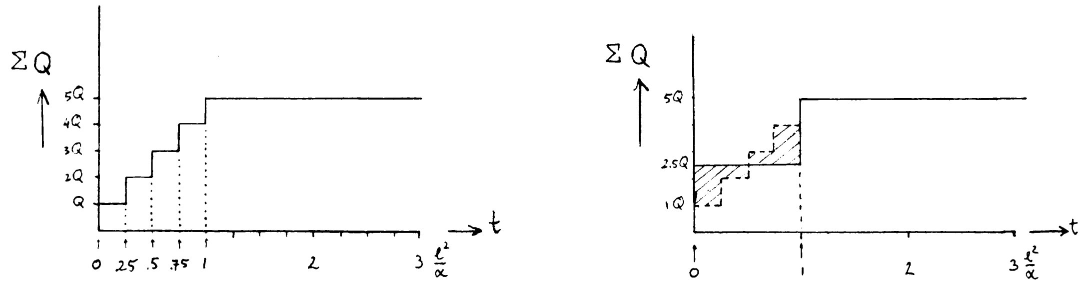

figure 6.20. *Reducing the number of elements*

The error introduced by the replacement is negligible if the potential or discharge is being calculated sufficiently later than the starting time of the last replaced element. In figure 6.21 the differences between the potentials for five and two wells are given as a function of time for four points. The strengths of the five and two wells are shown in figure 6.20. In figure 6.22 contours are shown at a time, equal to three times the last starting time.

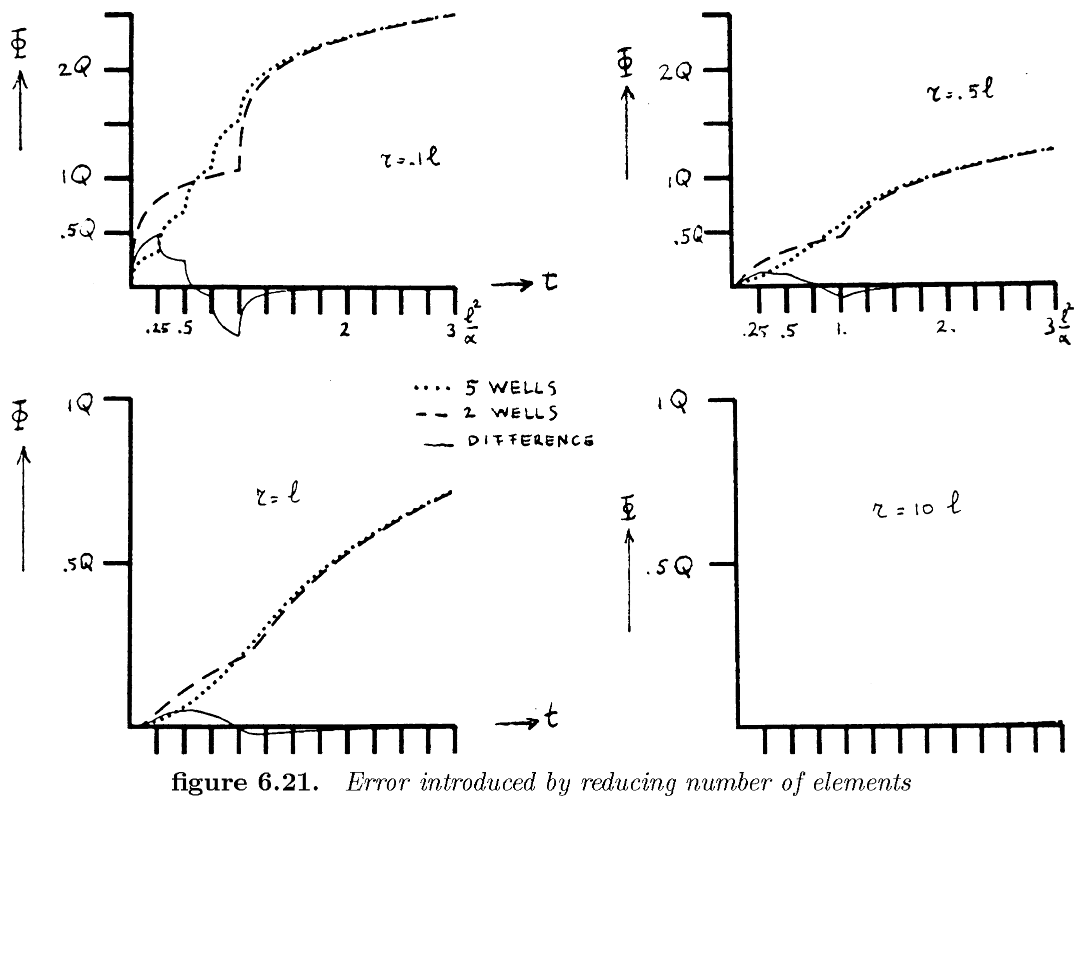

figure 6.21. *Error introduced by reducing number of elements*

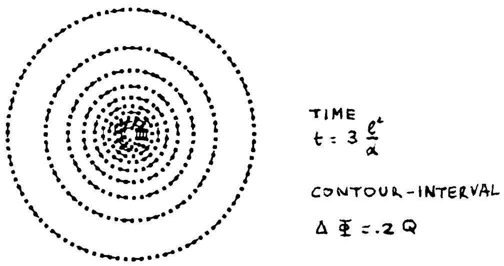

figure 6.22. *Contours due to five wells and two equivalent wells*

A second method of reducing computational times is to replace transient elements of order one at large times by their limits for the time becoming infinite (these are (4.27) for a well, (4.61) for a line-sink, (4.103) for an area-sink and the corresponding steady elements for a line-doublet.

In the following a potential is considered, that consists of the potentials (4.16) of m wells of degree zero. The wells all have the same location (x_w, y_w). The discharge Q_j of well j starts at time t_j, so that the potential is equal to

$$\begin{aligned}\Phi &= \sum_{j=1}^{m} \underset{w0}{\Phi}{}^{j} \\ &= \sum_{j=1}^{m}[-\frac{\underset{0}{Q}_j}{4\pi}\mathrm{E}_1(\frac{(x-x_w)^2+(y-y_w)^2}{4\alpha(t-t_j)})]\end{aligned} \tag{6.63}$$

If the current time t is so much larger then every starting time t_j that the limit (4.27) can be used then

$$\begin{aligned}\Phi &\simeq \sum_{j=1}^{m}[\frac{\underset{0}{Q}_j}{4\pi}\{\ln((x-x_w)^2+(y-y_w)^2)-\ln(4\alpha(t-t_j))+\gamma\}] \\ &= \sum_{j=1}^{m}[\frac{\underset{0}{Q}_j}{4\pi}\{\ln((x-x_w)^2+(y-y_w)^2)-\ln(4\alpha(t-t_j))\}]+\sum_{j=1}^{m}\underset{0}{Q}_j\frac{\gamma}{4\pi} \qquad t>>t_j\end{aligned} \tag{6.64}$$

which can be written as

$$\Phi \simeq \frac{\sum_{j=1}^{m}\underset{0}{Q}_j}{4\pi}\{\ln((x-x_w)^2+(y-y_w)^2)-\ln(4\alpha)+\gamma\}-\ln(\prod_{m}^{j=1}[(t-t_j)^{\underset{0}{Q}_j}]) \qquad t>>t_j \tag{6.65}$$

The evaluation of this expression requires much less computational effort than (6.63).

using elements of degree one or higher

In the above discussion line-sinks of degree zero were used. Therefor the discharge of the boundary segment is piecewise constant with discontinuities at the times of solving (see figure 6.2). Line-sinks of degree one ((4.68)) could have been used in stead. Their strength would have been chosen equal to zero at their starting time (see figure 6.23).

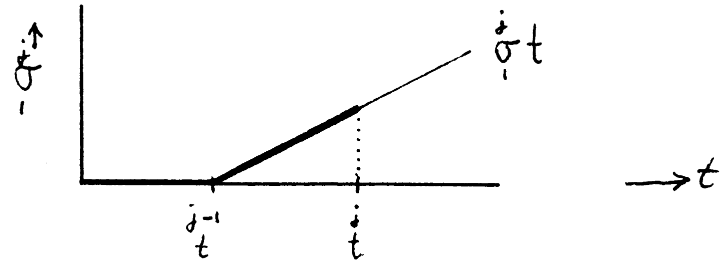

figure 6.23. *Strength of line-sinks of degree one for solving*

The number of equations in each time step would have been the same as with using line-sinks of degree one. The computational effort would not have been much more (compare (4.56) and (4.68)) per time step, while a larger size of the time steps could have been chosen. Moreover, the resulting distributed discharge at the boundary segment would have been continuous (see figure 6.24).

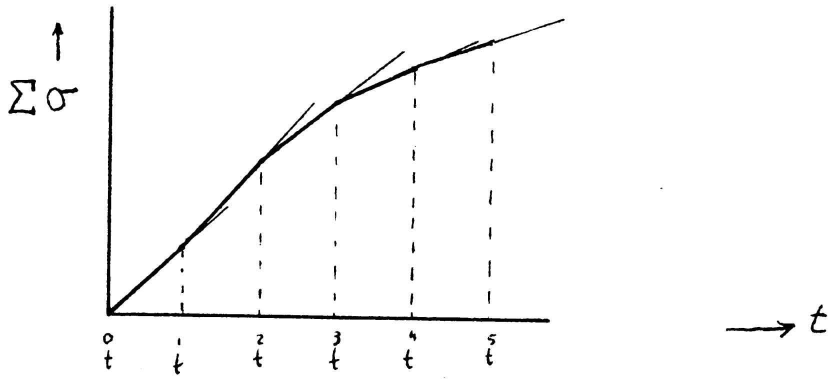

figure 6.24. *Summed discharge using line-sinks of degree one*

Another possibility is to use a line-sink of a high degree n. In that case the problem could have been solved at once, as was mentioned in **application of line-sinks of order one and degree** n in chapter 4. Then the system of equations would have been a lot larger. In stead of the equation for the unknown strength of the one new line-sink in every time step, there would have been a system of n equations of conditions at n different times at which the condition at the center of the boundary segment would have been met exactly. In modeling realistic problems the number of equations would soon be prohibitively large, if this method of solving were used.

7. comparisons with exact solutions

In this chapter five cases of flow are modeled with analytic elements. The modeling was done by means of a computer program. The results are compared with the exact solutions that were taken from Carslaw and Jaeger (1986).

The computer program is based on the model described in chapter 6. It contains wells and line-sinks with given strengths and line-segments with specified heads. The wells and line-sinks are of degree one. The program was written in ANSI FORTRAN-77 and runs on micro-computers under DOS. The size is 144 kilobytes, with a capacity of 33 equations and 5000 line-sinks, 50 wells and 100 boundary segments. The input can either be read from a file or entered inter-actively. The output can be graphical and numerical. Contour plots and graphs of the head can be generated on the screen. The strengths of the elements and values of the head are example of the options of written output.

The first four problems are cases of one-dimensional flow in a semi infinite aquifer. The flow is parallel to the x axis. The potential is defined for $x \geq 0$. Either the discharge or the potential is prescribed at the boundary $x = 0$. The fifth problem involves two-dimensional flow. It is the problem of a well near a long straight canal. The y axis chosen along the edge of the canal and the well is located on the positive x axis. Thus flow in the half plane $x \geq 0$ is modeled for all five problems. The aquifer diffusivity is equal to α and the potential is equal to the constant Φ_0 at time $t = 0$, the time that the transient effects start

$$\Phi\big|_{t\leq 0} = \Phi_0 \tag{7.1}$$

The condition at infinity requires that the potential does not change in the limit for $x \to \infty$

$$\lim_{x\to\infty} \Phi = \Phi_0 \tag{7.2}$$

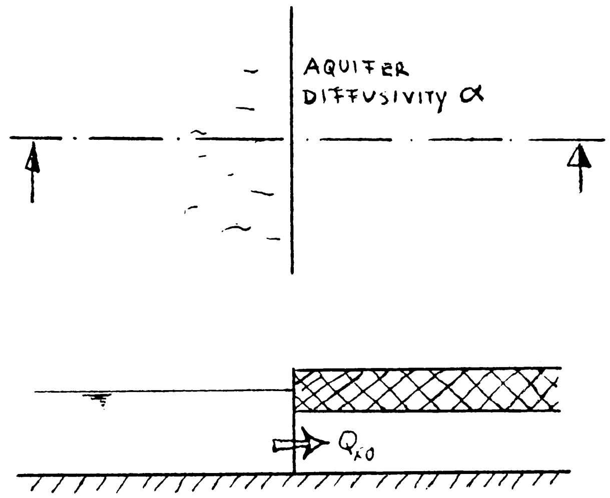

figure 7.1. *Flow due to discharge of degree zero at $x = 0$*

In the first of the one-dimensional problems the discharge is prescribed at the boundary $x = 0$. It starts at time zero and is equal to the constant value $\underset{0}{Q_x}$ from that moment on

$$Q_x\big|_{x=0} = \begin{cases} 0 & t \leq 0 \\ \underset{0}{Q_x} & t > 0 \end{cases} \tag{7.3}$$

The exact solution to this problem is given by (2.9.7) in Carslaw and Jaeger (1986). The potential can be written as

$$\Phi = \Phi_0 + \underset{0}{Q_x}[\frac{2\sqrt{\alpha t}}{\sqrt{\pi}}e^{-\frac{x^2}{4\alpha t}} - x\mathrm{erfc}(\frac{x}{2\sqrt{\alpha t}})] \tag{7.4}$$

The discharge is equal to

$$Q_x = \underset{0}{Q_x}\mathrm{erfc}(\frac{x}{2\sqrt{\alpha t}}) \tag{7.5}$$

The problem has been modeled with a very long line-sink of order one and degree zero (see figure 7.2). The flow is two-dimensional around a line-sink, but the equipotentials are practically straight close to its center. This is illustrated in figure 7.2. The line-sink has a length equal to one hundred times the length of the sides of the square in which the equipotentials are shown. Calculated potentials both at the mid-point of the line-sink and at the center of the square are compared with the potentials (7.4) for the exact solution in figure 7.3. The slight deviation of the curves for $x = 0$ is caused by the fact that the potential is not calculated exactly at the line-sink, but at a slight off-set, at $x = .02l$.

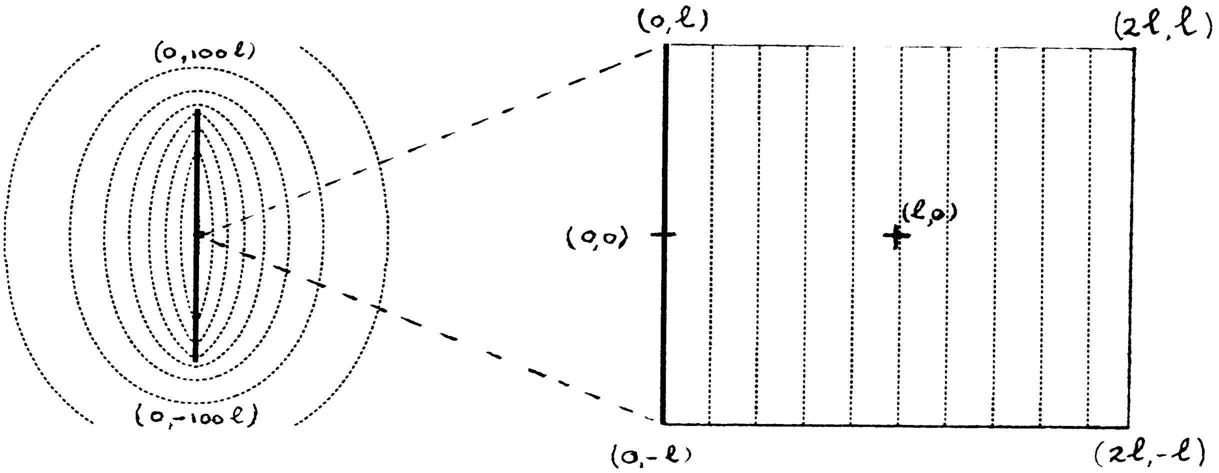

figure 7.2. *Model of flow due to discharge of degree zero at $x = 0$*

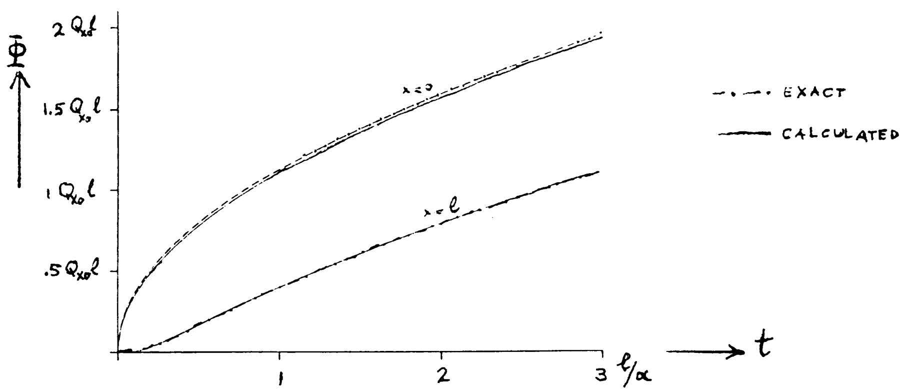

figure 7.3. *Calculated potential compared with exact solution to flow due to discharge of degree zero at $x = 0$*

The second problem is very similar to the first problem. The only difference is that the boundary condition (7.3) is replaced by

$$Q_x\Big|_{x=0} = \begin{cases} 0 & t \leq 0 \\ \underset{1}{Q}_x t & t > 0 \end{cases} \tag{7.6}$$

The exact solution is (Carslaw and Jaeger, 1986, equation 2.9.7)

$$\Phi = \Phi_0 + \underset{1}{Q}_x t\Big[\frac{4\sqrt{\alpha t}}{3\sqrt{\pi}}(1+\frac{x^2}{4\alpha t})e^{-\frac{x^2}{4\alpha t}} - (x+\frac{x^3}{6\alpha t})\mathrm{erfc}(\frac{x}{2\sqrt{\alpha t}})\Big] \tag{7.7}$$

$$Q_x = \underset{1}{Q}_x t\big[(1+\frac{x^2}{2\alpha t})\mathrm{erfc}(\frac{x}{2\sqrt{\alpha t}}) - \frac{x}{\sqrt{\pi\alpha t}}e^{-\frac{x^2}{4\alpha t}}\big] \tag{7.8}$$

The problem is modeled with a long line-sink of degree one. The potentials, obtained with a line-sink of degree one, at the centers both of the line-sink and the square in figure 7.2 are compared with the exact solution (7.7) in figure 7.4. The difference between the curves for $x = 0$ is again due to the fact that the potential in the program is actually calculated at $x = .02l$.

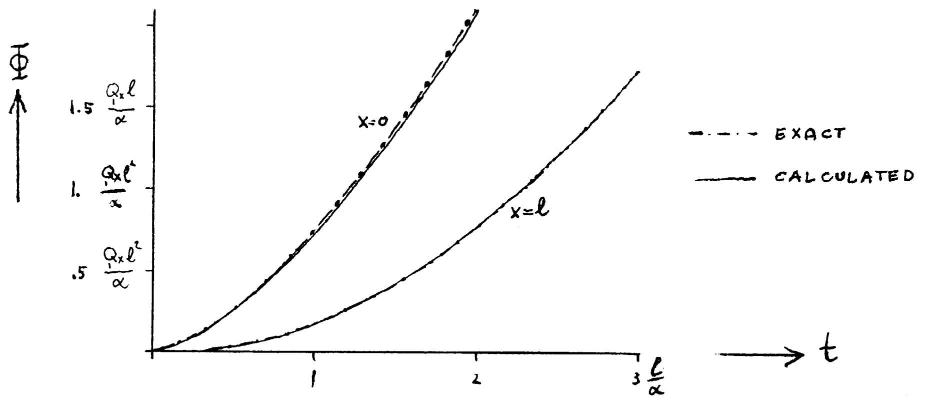

figure 7.4. *Calculated potential and potential for exact solution to flow due to a discharge of degree one at* $x = 0$

The third and fourth problem have boundary conditions in terms of the potential in stead of the discharge. The boundary is represented by line-segments at which the potential is specified.

For the third problem the potential at the boundary is raised instantaneously at time $t = 0$ to the value Φ_1

$$\Phi\Big|_{x=0} = \begin{cases} \Phi_0 & t \leq 0 \\ \Phi_1 & t > 0 \end{cases} \tag{7.9}$$

The exact solution to this problem (7.9) is given by equation (2.5.2) in Carslaw and Jaeger (1986).

$$\Phi = \Phi_0 + (\Phi_1 - \Phi_0)\mathrm{erfc}(\frac{x}{2\sqrt{\alpha t}}) \tag{7.10}$$

The discharge is equal to

$$Q_x = (\Phi_1 - \Phi_0)\frac{e^{-\frac{x^2}{4\alpha t}}}{\sqrt{\pi\alpha t}} \tag{7.11}$$

The problem was modeled with five line-segments with the potential specified at their centers (see figure 7.5). The problem has been solved stepping forward in time as has been described in chapter 6. At each time step a line-sink of order one and degree zero is added at each line-segment.

In the square in figure 7.5 the flow is practically one-dimensional. The potentials at (0, 0) and (1, 0), the middle of the square, are compared with the exact solution (7.10). The time step in figure 7.6 is ten times the step of figure 7.7. The discharges at $x = 0$ are compared with (7.11) in figure 7.8. The curves will coincide in the graphs in figures 7.6, 7.7 and 7.8 if the time-steps in the beginning are made smaller.

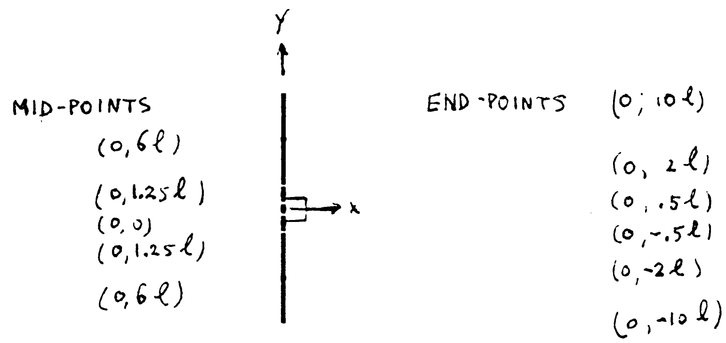

figure 7.5. *Model of flow due to sudden change in potential at $x = 0$*

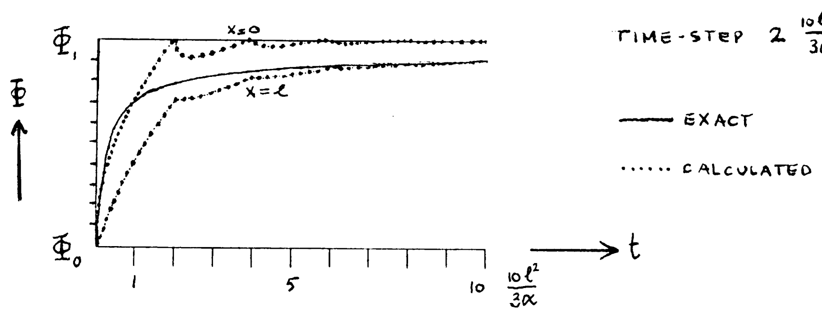

figure 7.6. *Calculated potential compared with exact solution to flow due to sudden change in potential at $x = 0$*

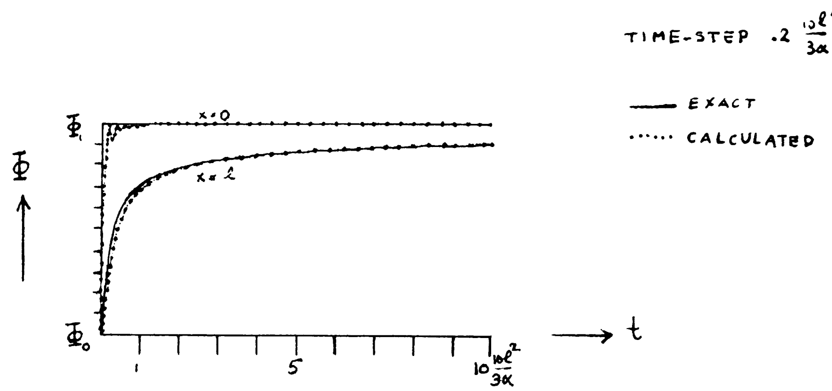

figure 7.7. *Calculated potential compared with exact solution to flow due to sudden change in potential at $x = 0$*

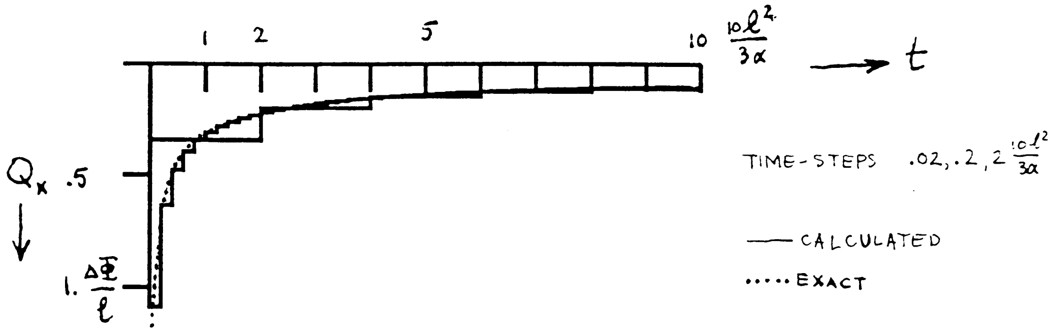

figure 7.8. *Calculated discharge compared with exact solution to flow due to sudden change in potential at* $x = 0$

In the fourth problem, the potential at the boundary rises lineary in time, starting at $t = 0$

$$\Phi\Big|_{x=0} = \begin{cases} \Phi_0 & t \leq 0 \\ \Phi_0 + \Phi_2 t & t > 0 \end{cases} \tag{7.12}$$

The exact solution is given by (2.5.4) in Carslaw and Jaeger (1986). The potential is equal to

$$\Phi = \Phi_0 + \Phi_2 t\big[(1 + \frac{x^2}{2\alpha t})\mathrm{erfc}(\frac{x}{2\sqrt{\alpha t}}) - \frac{x}{\sqrt{\pi \alpha t}} e^{-\frac{x^2}{4\alpha t}}\big] \tag{7.13}$$

The corresponding discharge is

$$Q_x = \Phi_2\big[-\frac{x}{\alpha}\mathrm{erfc}(\frac{x}{2\sqrt{\alpha t}}) + \frac{2\sqrt{t}}{\sqrt{\pi\alpha}} e^{-\frac{x^2}{4\alpha t}}\big] \tag{7.14}$$

The problem has been modeled using the same five line-segments as in the previous problem (see figure 7.5). The potentials that are specified at the line-segments are given new values for each solution, to reflect the linealy increasing potential. The potential at the center of the square in figure 7.5 is compared with the potential for the exact solution (7.13) in figure 7.9. The discharge at $x = 0$ is compared with (7.14) in figure 7.10. A large time-step was used so that the errors are clearly visible. This does not cause the errors to become increasingly large in time.

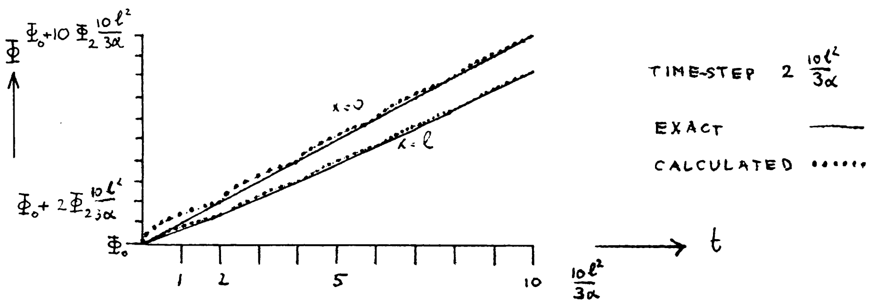

figure 7.9. *Calculated potential and potential for exact solution to flow due to linear increase of the potential at $x = 0$*

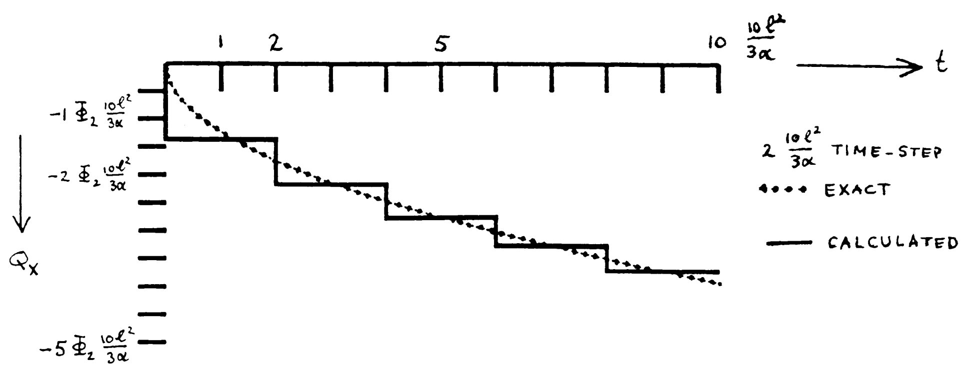

figure 7.10. *Calculated discharge and discharge for exact solution to flow due to linear increase of the potential at $x = 0$*

The fifth problem is the problem of a well near a long, straight canal (see figure 7.11). The water level in the canal is constant, so that it is an equipotential for the neighboring groundwater flow. The equipotential coincides with the y-axis. The well is of degree zero (4.16), is located at $(2l, 0)$, and starts pumping at time zero

$$Q_0 = \begin{cases} 0 & t \leq 0 \\ Q_w & t > 0 \end{cases} \tag{7.15}$$

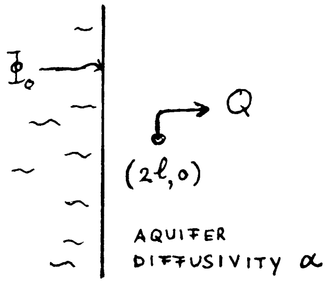

figure 7.11. *Problem with a well near a canal*

The exact solution is obtained by using the method of images. A second well with the opposite discharge of (7.15) is put at a point which is the mirror image of the location of the first well (see figure 7.12) with respect to the equipotential. The potential is given by

$$\begin{aligned} \Phi = \Phi_0 &- \frac{Q_w}{4\pi} \mathrm{E}_1\left(\frac{(x-2l)^2+y^2}{4\alpha t}\right) \\ &+ \frac{Q_w}{4\pi} \mathrm{E}_1\left(\frac{(x+2l)^2+y^2}{4\alpha t}\right) \end{aligned} \tag{7.16}$$

where the potential for a well (4.16) has been used. The potential (7.16) is indeed equal to Φ_0 at the y-axis as can be seen from simple substitution of $x = 0$. The discharge in x-direction is equal to

$$\begin{aligned} Q_x = &- \frac{Q_0}{2\pi} \frac{x-2l}{(x-2l)^2+y^2} e^{-\frac{(x-2l)^2+y^2}{4\alpha t}} \\ &+ \frac{Q_0}{2\pi} \frac{x+2l}{(x+2l)^2+y^2} e^{-\frac{(x+2l)^2+y^2}{4\alpha t}} \end{aligned} \tag{7.17}$$

where (4.18) has been used.

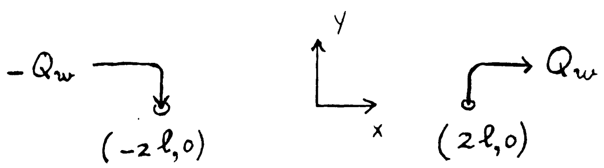

figure 7.12. *Exact solution for problem with a well near a canal*

The problem has been modeled similarly to the third and fourth problem (see figure 7.13).

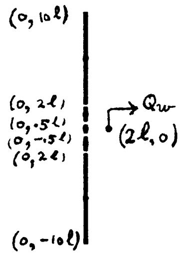

figure 7.13. *Model for problem with a well near a canal*

The potential at a point in between the well and the equipotential (see figure 7.11) is compared with the exact solution (7.16) for two different sets of time steps. The time step in figure 7.14 is ten times as large as the time step in figure 7.15. In figure 7.16 the discharge in x-direction at the point $(0, 0)$ obtained with the first set of time steps with the program is compared with the exact solution (7.17). The same large step is used for the solution that is contoured in figure 7.17. Equipotentials from the computer program and the exact solution are plotted in one plot in figure 7.17. The time of the contours is at the end of the first time step and at two fifths of the time step, when the error at the centers of the line-segments are close to the maximum values.

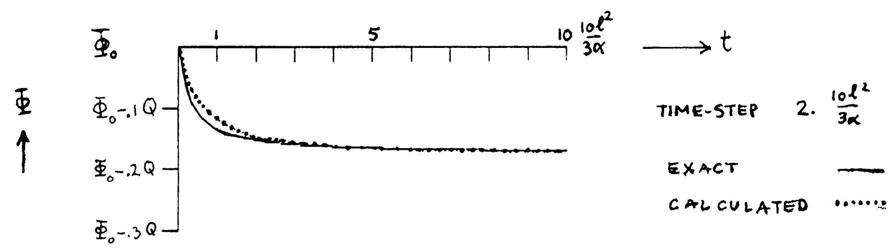

figure 7.14. *Calculated potential and potential for exact solution of problem with a well near a canal*

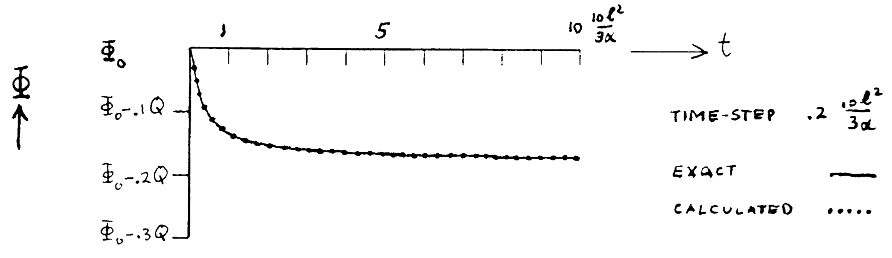

figure 7.15. *Calculated potential and potential for exact solution of problem with a well near a canal*

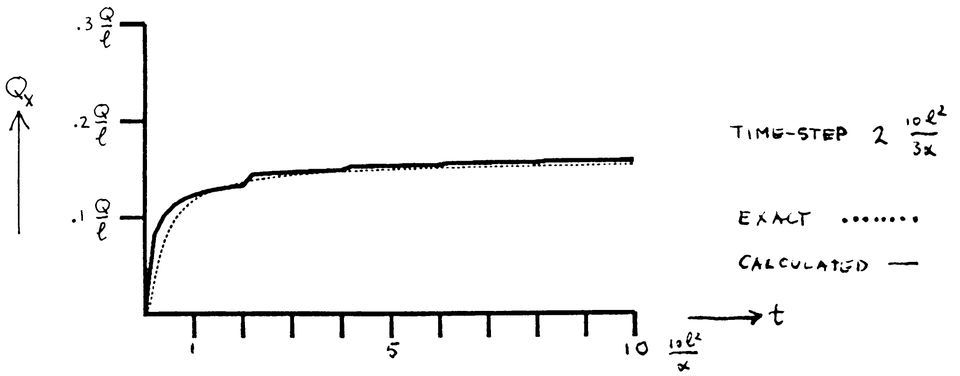

figure 7.16. *Calculated discharge and discharge for exact solution of problem with a well near a canal*

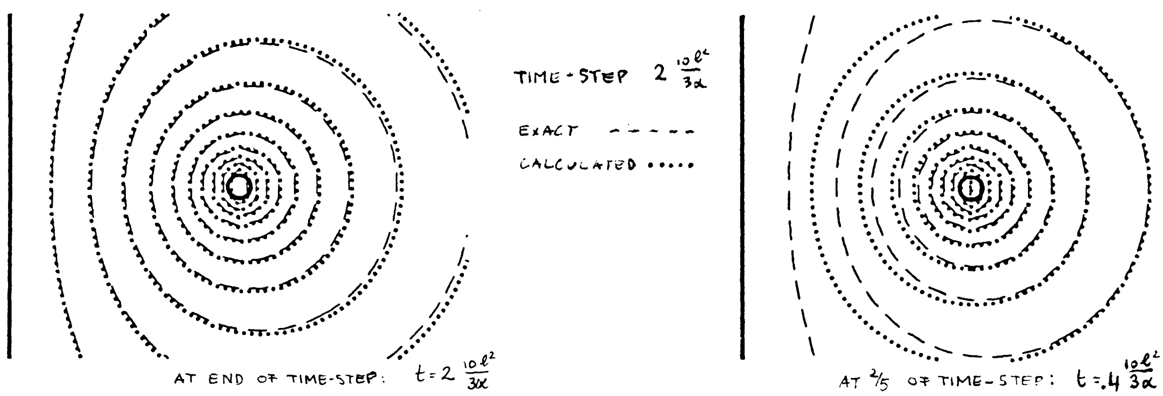

figure 7.17. *Calculated equipotentials and equipotentials for exact solution of problem with a well near a canal*

8. application of computer program

In this chapter an application of the analytic element method for transient flow, as discussed in the previous chapters, is given. A hypothetical case of groundwater flow will be modeled.

The computer program TS, that is used for the modeling was written for a project funded by the Legislative Commission on Minnesota Resources. It is to be used in conjunction with a separate program for steady flow, SLAEM.

The initial steady state is modeled with SLAEM. This program has the option to store a solution on disk. The solution is read by the program TS. It automatically places a transient line-segment with a specified head at the location of each steady line-sink with a specified head. It is possible to change the values of the specified heads. This can be done immediately or after one or more time-steps have been performed. More line-segments can be added as well as transient wells, line-sinks and area-sinks with given strength of degree zero. The program also contains the far-field functions, that were discussed in chapter 5. The method of solving in time-steps was outlined in chapter 6. The program can generate contour plots of the head, plots of path-lines and a variety of numerical output.

Both programs SLAEM and TS are written in ANSI FORTRAN-77 and run on a MC68020 co-processor board installed in a micro-computer. The program TS with a capacity for 100 equations, 50 area-sinks, 50 wells and 1000 line-sinks takes up about 30 percent of the 1 Mbyte of memory that is available on the board.

The hypothetical problem (see figure 8.1) is centered around a number of agricultural fields. The fields are located in an area between two forks of a river. The level of the river is taken to be constant during the entire year. The yearly cycle of the precipitation is simplified to a half year of zero infiltration and a half year of constant infiltration. The fields are irrigated during a part of the dry season. The water for the irrigation is pumped up from wells that are located nearby. The total infiltration of the irrigation is taken to be equal to the extraction of the wells from the aquifer. All quantities will be expressed in terms of meters [m] and days [day].

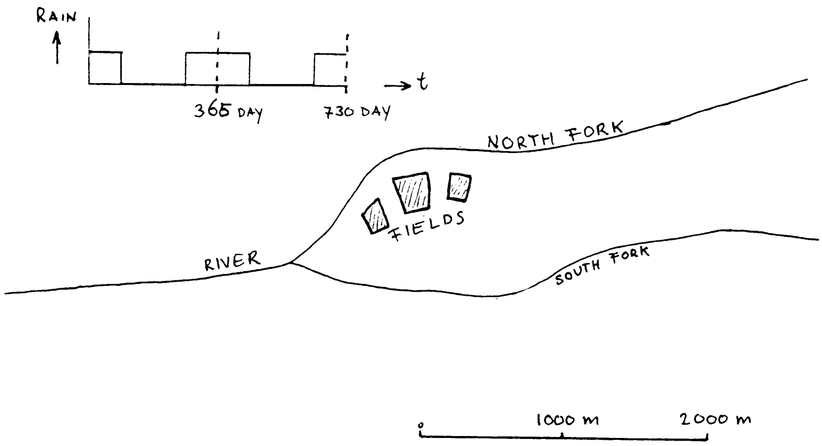

figure 8.1. *Hypothetical problem*

The modeling will be carried out in steps. After the data have been specified, an average steady state will be determined. Next transient modeling will be done which includes the seasonal variation of the precipitation. Finally the irrigation and extraction will be added.

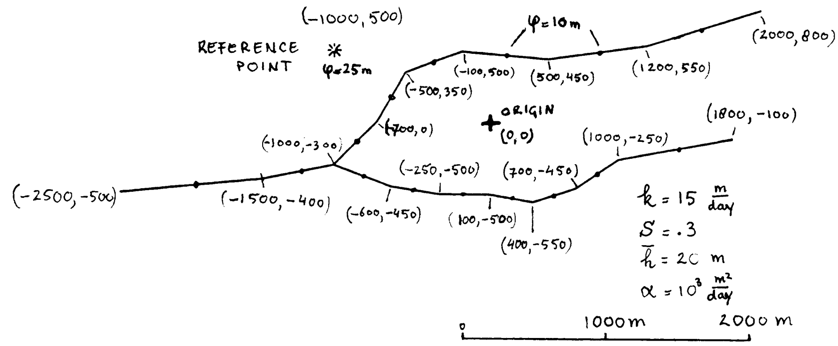

figure 8.2. *Geometry of problem for modeling*

The geometry as it is approximated for the modeling is shown in figure 8.2. The river is represented by 15 straight line-segments. The aquifer is unconfined with a hydraulic conductivity of $k = 15$ m/day. The storativity S is equal to the porosity $\nu = .3$. The average saturated thickness is set equal to $\bar{h} = 20$m so that the aquifer diffusivity (2.19) has a value of $\alpha = 1000$ m^2/day. The river is in direct contact with the aquifer, so that the head at the river is equal to the level of the water in the river, which is 10 m. The extraction that represents the precipitation is equal to E =0 during one half year and equal to $E = -.1$ m/day the next half year. The fields are irrigated for a period of hundred days in each dry season with an extraction rate of $E = -.1$ m/day.

The dimensions of the problem and the values of most parameters are fairly realistic, except for the rain and the irrigation. The values of both the rain and the irrigation have been exaggerated, so that their influence on the groundwater flow is clearly visible.

The area shown in figure 8.2 contains all elements that will be used. A smaller area will be considered to evaluate the results of the modeling. This area is indicated in figure 8.3.

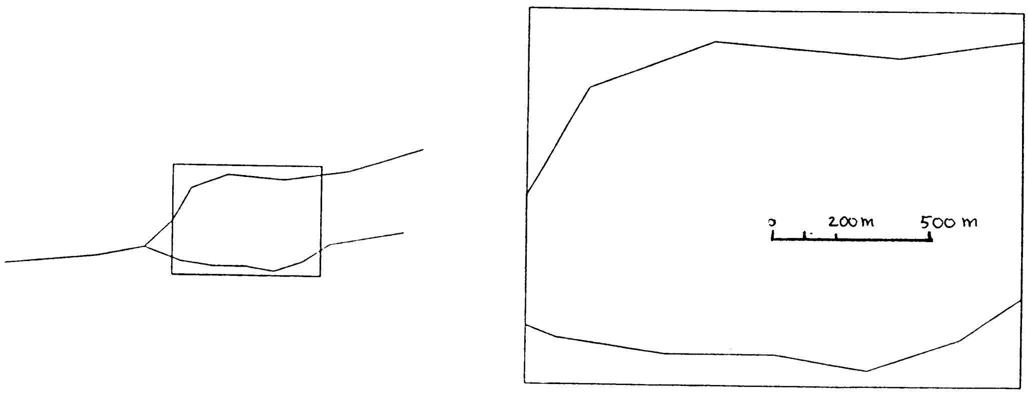

figure 8.3. *Area used for output of the model*

First the steady state model will be discussed. A steady line-sink is placed at each straight line-segment of the river (see figure 8.3). The strength is determined by the condition that the head at the center is equal to $\varphi = 10$m. The yearly averaged rain is represented by a steady uniform extraction function. For the reference-point the location $(-1000, 500)$ has been chosen with a head $\varphi = 25$m.

After the solution had been made the resulting heads were contoured. The contour plot is given in figure 8.4.

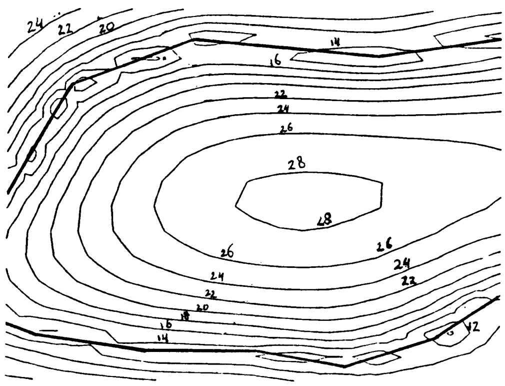

figure 8.4. *Contour plot of the heads for the steady state*

Secondly the model for the transient state with the seasonally varying infiltration will be presented. Transient area-sinks are superimposed on the steady state to change the constant average rain into seasonally varying rain. The area-sinks have the same location and cover the entire model area (see figure 8.5). The starting times are different (see figure 8.6). These are chosen such that the transient model starts in the middle of a period with rain, so that the initial situation is close to the transient situation at that time if the transient flow had started a long time back .

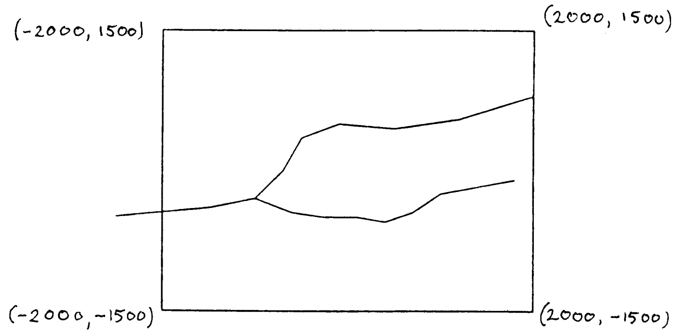

figure 8.5. *Area-sinks used for the varying precipitation*

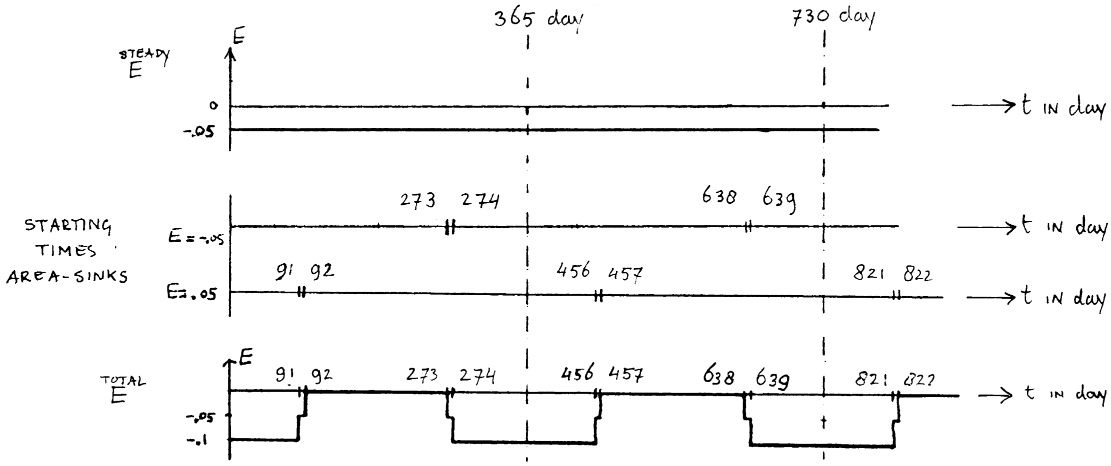

figure 8.6. *Seasonally extraction of area-sinks and steady rain*

The river is modeled by transient line-sinks. At selected time intervals new transient line-sinks are added at the straight sections of the river, on top of the steady line-sinks of the initial steady state. The strengths of the line-sinks that started at the beginning of the time-interval are determined from the condition that the head at the center of the line-sink at the end of the time-interval is equal to the river-level $\varphi = 10$m.

The far-field functions of chapter 5 are not needed, since fluctuations around an average situation are modeled.

A solution was made with time-intervals $\Delta t_{solve} = 30$day. Plots with contours of the heads at several different times are given in figure 8.7.

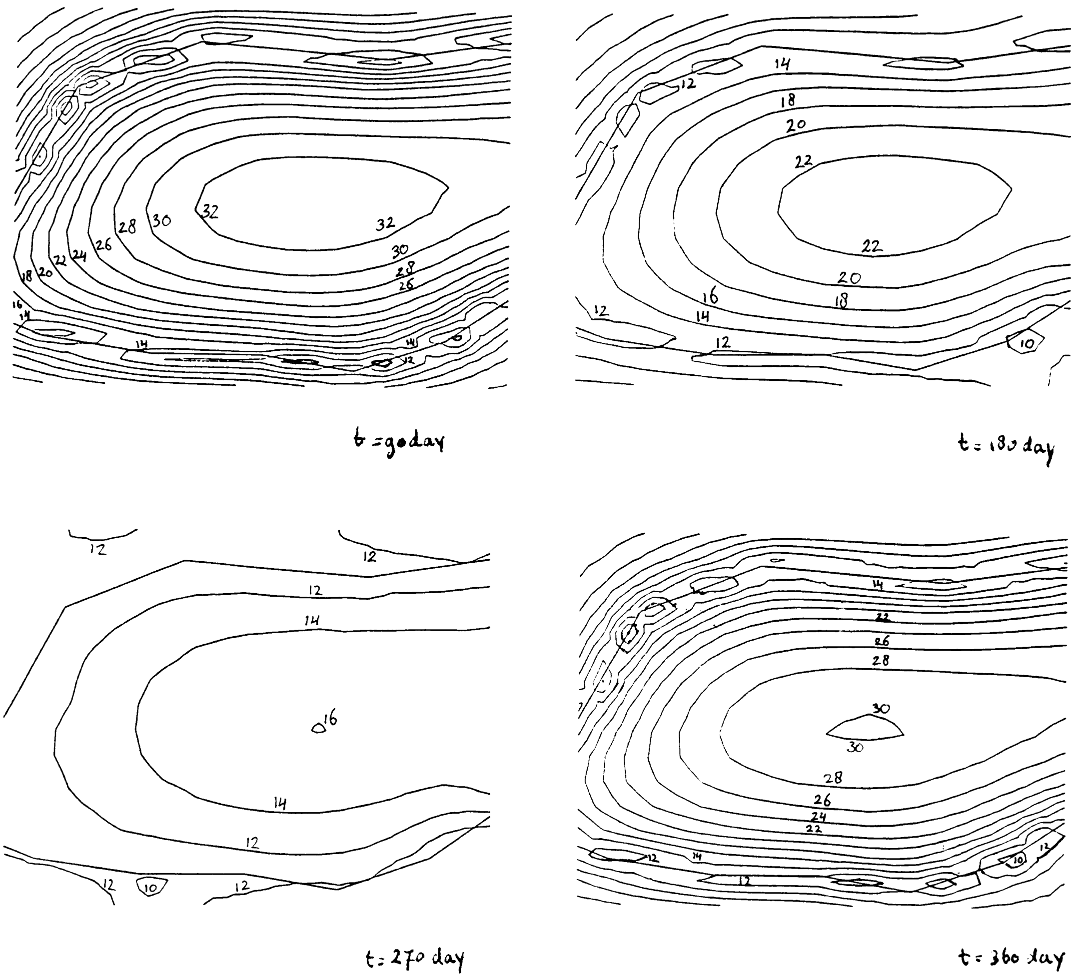

figure 8.7. *Contour plots of the head for the transient state without irrigation*

Thirdly the irrigation and extraction will be added to the transient model. Two periods are chosen for the irrigation and extraction. The first one toward the end $t = 273$day of the first dry half year from $t = 150$day to $t = 250$day. The second one toward the end $t = 638$day of the second dry period from $t = 525$day to $t = 625$day. The periods are indicated in figure 8.8 in the graph for the extraction representing the precipitation versus time.

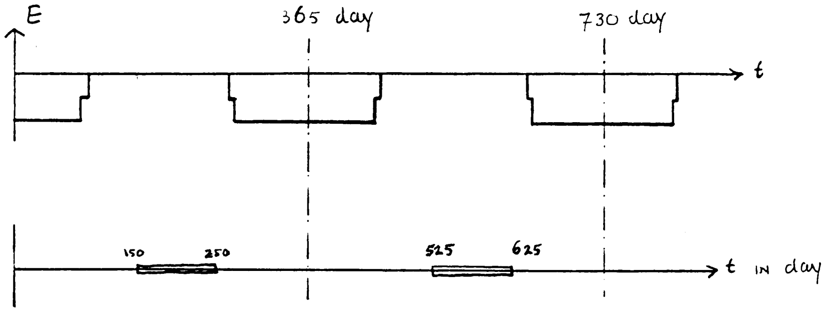

figure 8.8. *Periods of irrigation*

In the first period the water that is irrigated on the three fields is pumped up from the aquifer with 5 wells with each a discharge of $Q = 3174.9\text{m}^3/\text{day}$. In the second period all irrigation water is pumped up with one single well pumping $Q = 15874.5\text{m}^3/\text{day}$. In both periods the total discharge of the wells is taken to be equal to the total infiltration of the area-sinks that model the irrigation (see figure 8.9).

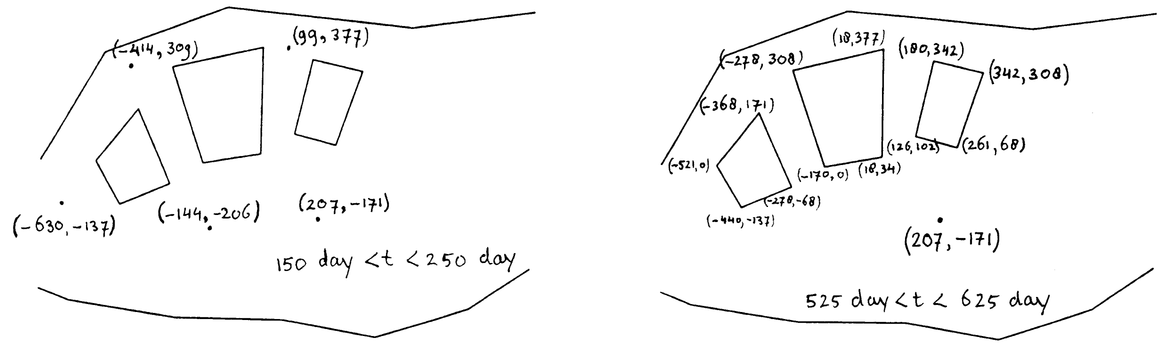

figure 8.9. *Configuration of irrigation and extraction*

The river is modeled in the same way as for the transient state with only the seasonal varying rain. Again no far-field functions are needed since the net discharge of the transient elements fluctuates around zero. The solution was made with time intervals of $\Delta t_{solve} = 25\text{d}$. Contours during the period of irrigation and extraction are given in figure 8.10.

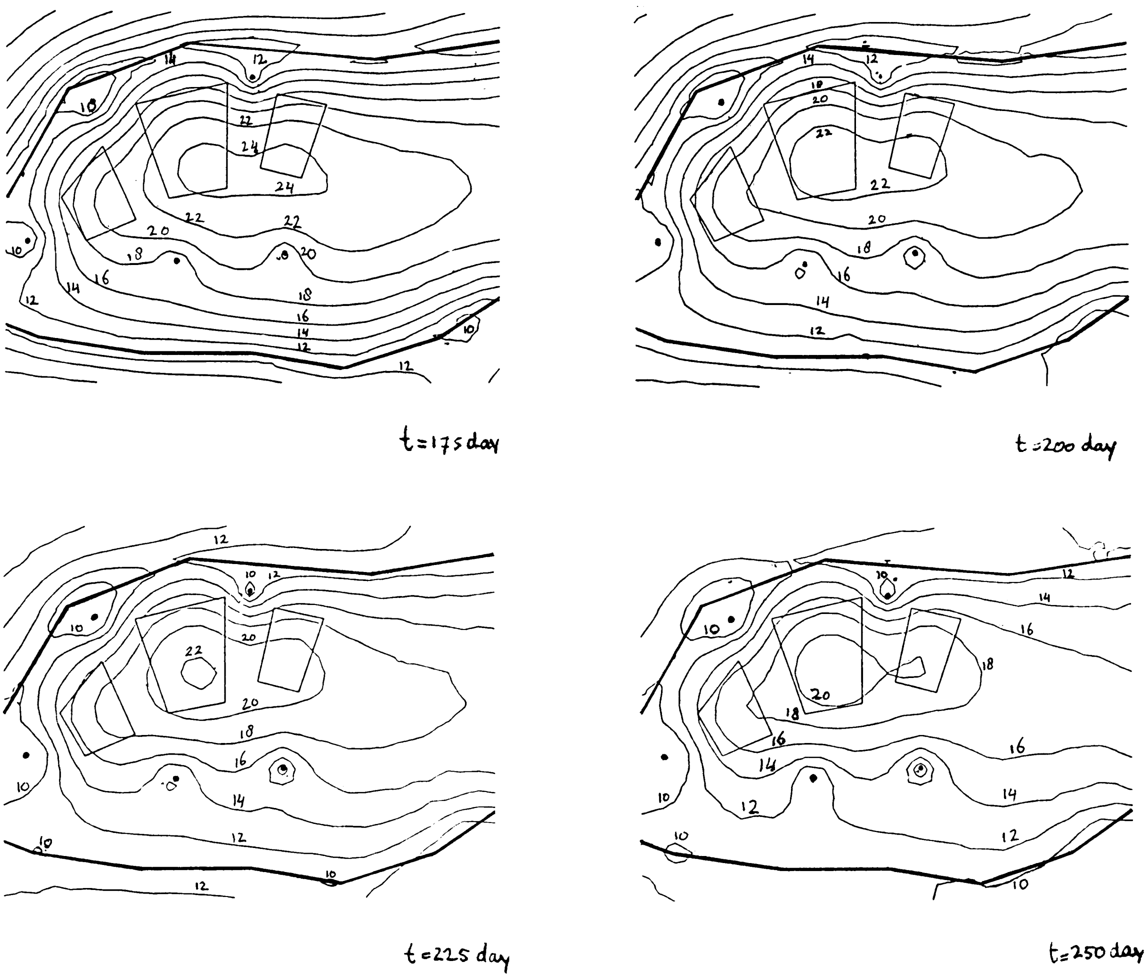

figure 8.10. *Contour plots of the head in the transient model with irrigation and extraction using five wells*

Contours during the second period of irrigation and extraction are given in figure 8.11. The contours indicate that the level of the head becomes equal to zero at some distance from the well. This is physically impossible. It means that the discharge of the well is larger than is possible in reality. So it is not feasible to use just one well to pump up the water for the irrigation.

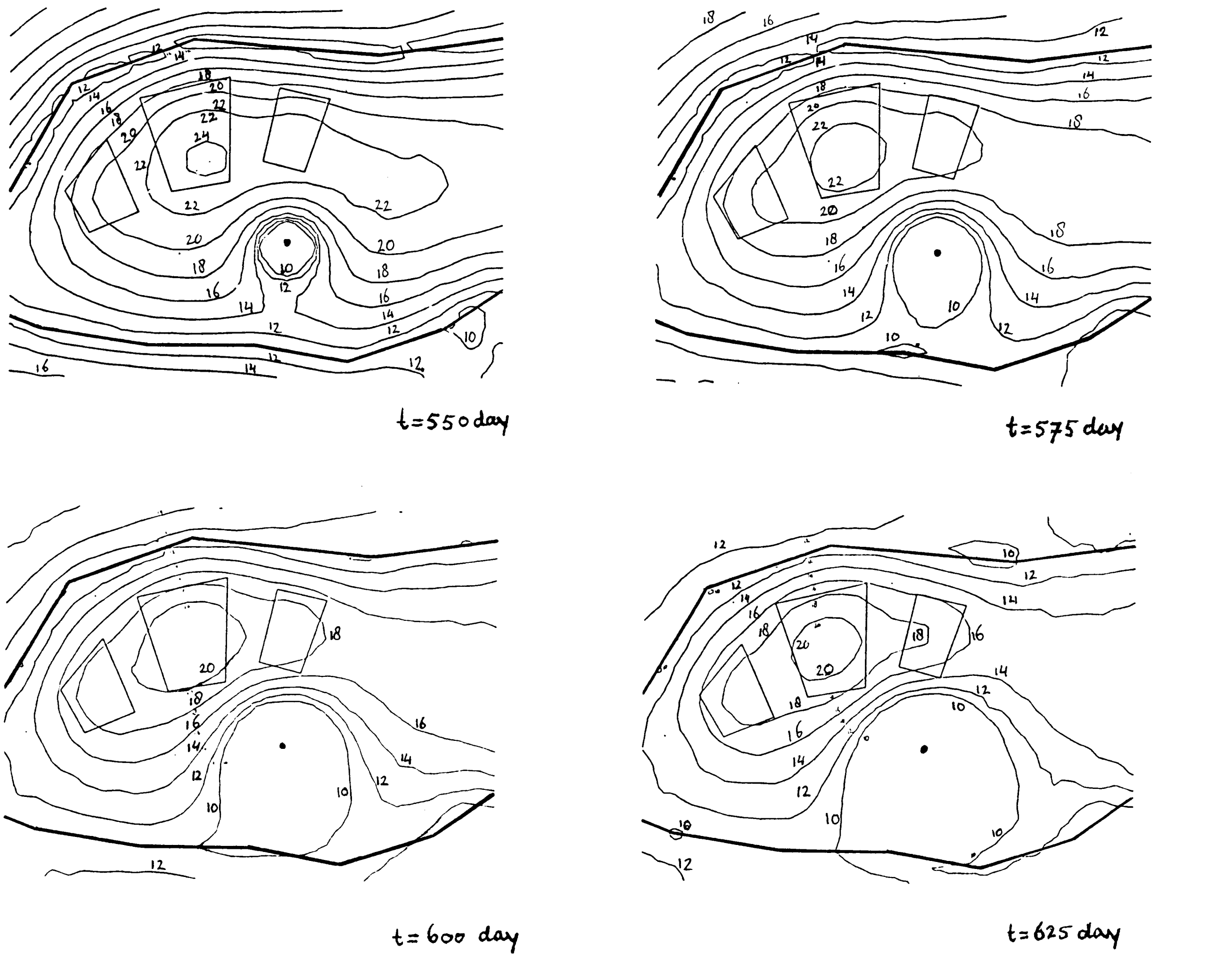

figure 8.11. *Contour plots of the head in the transient model with irrigation and extraction using one well*

9. concluding remarks

In this chapter some concluding remarks are given about the analytic elements for transient flow that have been derived in this thesis and their application to modeling groundwater flow.

Transient groundwater flow can be modeled accurately with the model that was described in chapter 6. The hydraulic head and the components of the discharge vector are known as functions of position and time in the model.

It is a useful extension to the analytic element method. Regional flow can be modeled accurately. To do the modeling efficiently more effort has to be put into reducing the computational times. Gains can be made in the functions of the elements and the evaluation of a number of elements at one location as they occur on the boundaries with unknown strength. The computer program, that was demonstrated in chapter 8, is too slow to be able to model practical regional flow problems interactively.

The adjustment of the far-field as presented in chapter 5 is not very elegant. If the expression for the Glover well or the ring-source could be extended so that it is valid in the entire plane, that one function would be the only one needed to adjust the far-field and the modeling would not be restricted by the radius R. Moreover the well could be used to create a dipole at infinity which would make it possible to change the uniform flow between the initial and the final steady state.

The model described in chapter 6 will be applicable to a wider range of regional groundwater flow problems if it is extended to include domains with an extraction, that depends on the head, and inhomogeneities in the aquifer parameters. The domains would be used to model leakage through the confining layers from the aquifer to lakes or rivers above the aquifer or to other aquifers (compare Strack(1988) for the steady equivalents). The varying extraction of the domains could be modeled by area-sinks with different starting times and unknown strengths. Inside an inhomogeneity, the parameters in the definitions for the potential and the aquifer diffusivity have different values, which causes a discontinuity in the potential and the diffusivity across the boundary of the inhomogeneity (see Strack(1988) for the modeling of inhomogeneities in steady flow). In unconfined flow, regions with a significantly different average saturated thickness could be modeled as an inhomogeneity with only a different value of the aquifer diffusivity.

references

M. Abramowitz, I.A. Stegun (1972), Handbook of Mathematical Functions, ninth printing, Dover, New York NY, USA, 1972.

M.A. Abu-Zied, V.H. Scott (1963), Nonsteady Flow for Wells with Decreasing Discharge, Proceedings of the American Society of Civil Engineers, Journal of the Hydrology division, vol 89, no HY3, 119-132, 1963.

J. Barkley Rosser (1948), Theory and Application of $\int_0^z \exp(-x*x)dx$ and $\int_0^z \exp(-p*p*y*y)dy \int_0^y \exp(-x*x)dx$, Mapleton House, Brooklyn NY, USA, 1948.

R.J. Barnes, O.D.L. Strack (2003), A transient area-sink composed of line elements, paper for the ICAEM conference that was to be held in April 2003, but was cancelled, University of Minnesota, Minneapolis MN, USA, 2003.

J. Bear, A. Verruijt (1987), Modeling Groundwater Flow and Pollution, D. Reidel, Dordrecht, the Netherlands, 1987.

H.S. Carslaw, J.C. Jaeger (1986), Conduction of Heat in Solids, second edition, Clarendon Press, Oxford, 1986.

R.E. Glover (1974), Transient Groundwater Hydraulics, Department of Civil Engineering, Colorado State University, Fort Collins, Colorado, USA, 1974.

M.S. Hantush (1964), Drawdown Around Wells of Variable Discharge, Journal of Geophysical Research, vol 69, no 20, 1964.

C.E. Jacob, S.W. Lohman (1952), Nonsteady Flow to a Well of Constant Drawdown in an Extensive Aquifer, Transactions American Geophysical Union, vol 33, no 4, 1952.

B. Litkouhi, J.V. Beck (1982), Temperatures in Semi-Infinite Body Heated By Constant Heat Flux Over Half Space, in: Heat Transfer 1982 Proceedings of the Seventh International Heat Transfer Conference Muenchen, Fed.Rep. of Germany, vol 2 general papers, edited by: E. Hahne, K. Stephan, J. Straub, Hemisphere, Washington DC, USA, 21-27, 1982.

O.D.L. Strack (1986), Regional Groundwater Modeling by the Analytic Element Method, in: Advancements in Aerodynamics, Fluid Mechanics and Hydraulics, edited by R.E.A. Arndt, H.G. Stefan, C. Farrell, S.M. Peterson, ASCE, Minneapolis MN, USA, 1986.

O.D.L. Strack (1988), Groundwater Mechanics, Prentice Hall, Englewood Cliffs NJ, USA, to appear.

C.V. Theis (1935), The Relation Between The Lowering of the Piezometric Surface and the Rate and Duration of Discharge of a Well Using Ground-Water Storage, Transactions American Geophysical Union, Reports and Papers Hydrology, 1935.

H.F. Weinberger (1965), A First Course in Partial Differential Equations with Complex Variables and Transform Methods, Wylie, New York NY, USA, 1965.

C.R. Wylie, L.C. Barrett (1982), Advanced Engineering Mathematics, fifth edition, first printing, McGraw-Hill, Tokyo, Japan, 1982.

appendix A. the function $\int_{a\sqrt{\eta^2}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du$

In this appendix the function

$$\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du \tag{A.1}$$

is discussed.

The function is closely related to functions that are discussed by Barkley Rosser (1948) and Litkouhi and Beck (1982). Using the equality $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$ (equation 7.1.2 of Abramowitz and Stegun, 1972) equation (A.1) can be written as

$$\begin{aligned}
&\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du \\
&\quad= \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{1 - \mathrm{erf}(u\frac{\xi-\xi_2}{\eta}) - 1 + \mathrm{erf}(u\frac{\xi-\xi_1}{\eta})\}\,du \\
&\quad= \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erf}(u\frac{\xi-\xi_1}{\eta}) - \mathrm{erf}(u\frac{\xi-\xi_2}{\eta})\}\,du \\
&\quad= \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\,\mathrm{erf}(u\frac{\xi-\xi_1}{\eta})\,du - \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\,\mathrm{erf}(u\frac{\xi-\xi_2}{\eta})\,du
\end{aligned} \tag{A.1.1}$$

For the integrals in this equation the following change of variables is applied

$$\begin{aligned}
v &= u\frac{\xi-\xi_1}{\eta} & \qquad w &= u\frac{\xi-\xi_2}{\eta} \\
u &= v\frac{\eta}{\xi-\xi_1} & \qquad u &= w\frac{\eta}{\xi-\xi_2} \\
du &= \frac{\eta}{\xi-\xi_1}dv & \qquad du &= \frac{\eta}{\xi-\xi_2}dw
\end{aligned} \tag{A.1.2}$$

The change of variables results in

$$\begin{aligned}
&\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du \\
&\quad= \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\,\mathrm{erf}(u\frac{\xi-\xi_1}{\eta})\,du - \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\,\mathrm{erf}(u\frac{\xi-\xi_2}{\eta})\,du \\
&\quad= \int_{\frac{\xi-\xi_1}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-(\frac{\xi-\xi_1}{\eta})^2}\mathrm{erf}(v)\frac{\eta}{\xi-\xi_1}\,du - \int_{\frac{\xi-\xi_2}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-(v\frac{\xi-\xi_2}{\eta})^2}\mathrm{erf}(w)\frac{\eta}{\xi-\xi_2}\,du \\
&\quad= \frac{\sqrt{\pi}}{2}\Big[\frac{\eta}{\xi-\xi_1}\frac{2}{\sqrt{\pi}}\int_{\frac{\xi-\xi_1}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-(\frac{\xi-\xi_1}{\eta})^2}\mathrm{erf}(v)\,du \\
&\qquad\quad - \frac{\eta}{\xi-\xi_2}\frac{2}{\sqrt{\pi}}\int_{\frac{\xi-\xi_2}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-(v\frac{\xi-\xi_2}{\eta})^2}\mathrm{erf}(w)\,du\Big] \\
&\quad= \frac{\sqrt{\pi}}{2}\Big[\mathrm{H}(\frac{\xi-\xi_1}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}, \frac{\eta}{\xi-\xi_1}) - \mathrm{H}(\frac{\xi-\xi_2}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}, \frac{\eta}{\xi-\xi_2})\Big]
\end{aligned} \tag{A.1.3}$$

Where the definition of the function H in equation 12 of Litkouhi and Beck (1982) has been used:

$$\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2} \{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\, du \tag{A.1.4}$$

The function calls of H are reworked as:

$$\begin{aligned}
&\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2} \{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\, du \\
&=\frac{\sqrt{\pi}}{2}[\mathrm{H}(\frac{\xi-\xi_1}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}, \frac{\eta}{\xi-\xi_1}) - \mathrm{H}(\frac{\xi-\xi_2}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}, \frac{\eta}{\xi-\xi_2})] \\
&=\frac{\sqrt{\pi}}{2}[\mathrm{H}(\frac{\xi-\xi_1}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}, (\frac{\xi-\xi_1}{\eta})^{-1}) - \mathrm{H}(\frac{\xi-\xi_2}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}, (\frac{\xi-\xi_2}{\eta})^{-1})] \\
&=\frac{\sqrt{\pi}}{2}[\mathtt{LB82}(\sqrt{\frac{\eta^2}{4\alpha t}}, \frac{\xi-\xi_1}{\eta}) - \mathtt{LB82}(\sqrt{\frac{\eta^2}{4\alpha t}}, \frac{\xi-\xi_2}{\eta})] \\
&=\mathtt{IEER}(\sqrt{\frac{\eta^2}{4\alpha t}}, \frac{\xi-\xi_2}{\eta}, \frac{\xi-\xi_1}{\eta})
\end{aligned} \tag{A.1.5}$$

Where `LB82` and `IEER` denote the Real Functions used to code the approximation of the integral (A.1). The function `IEER` is called with the following arguments:

```
IF (ABS(RY).LT.RTOL) THEN
   IF (RX1.LT.RX2) THEN
      RX1T = RX1
      RX2T = RX2
   ELSE
      RX1T = RX2
      RX2T = RX1
   END IF
   IF (RX.LE.RX1T .OR. RX.GE.RX2T) THEN
      RVAL = 0.D0
   ELSE
      IF (RY.GE.0.D0) THEN
         RVAL = 1.7724538509055160272981 67D0                    ! SQRT(PI)
      ELSE
         RVAL =-1.772453850905516027298167D0                    ! -
SQRT(PI)
      END IF
      IF (RX2.LT.RX1) RVAL=-RVAL
   END IF
ELSE
   RXIN=0.5*ABS(RY)*SQRT(RP/RTIME)
   RA=(RX-RX2)/RY
   RB=(RX-RX1)/RY
   RVAL=RFIEER(RXIN,RA,RB)
END IF
```

The listing of the function and subsequent functions is given below. The functions are based on Litkouhi & Beck (1982) and Barnes & Strack (2003). Note that the variable RP is the inverse of the aquifer diffusivity α: RP= $1/\alpha$.

```
      DOUBLE PRECISION FUNCTION RFIEER(RX,RA,RB)
*     function returns the value of the integral of the e power of minus
```

```
*     x squared times the difference of the complementary error functions
*     of RA*x and RB*x evaluated from x=RX to x=infinity
      IMPLICIT NONE
      DOUBLE PRECISION RX, RA, RB
      DOUBLE PRECISION RSQRPI, RCUTFX
*                      Abramowitz & Stegun: page 3
      PARAMETER       (RSQRPI=1.772453850905516027298167D0,
     &                 RCUTFX=10.D0)
      DOUBLE PRECISION RFLB82
      IF (RX.GT.RCUTFX) THEN
*        for large x the integral is zero because the exp(-x*x) part of
*        the integrand becomes zero and the rest stays finite
         RFIEER=0.D0
      ELSE
*        the function can be expressed in terms of the funtion H
*        defined in eq.12 in Litkouhi and Beck (1982)
*        RFIEER(x,a,b)=.5/sqrt(pi)*[H(b*x,1/b)-H(a*x,1/a)]
*                     =.5/sqrt(pi)*[LB82(x,b) - LB82(x,a)]
*        without combining anything (which would speed things up in some
*        cases) this can be calculated as:
         RFIEER=.5D0*RSQRPI*( RFLB82(RX,RB)-RFLB82(RX,RA) )
      END IF
      RETURN
      END
```

The function LB82 which the above function IEER calls is listed below:

```
      DOUBLE PRECISION FUNCTION RFLB82(RXIN,RAIN)
*     RF.................L...........B.......82
*     function taken from Litkouhi and Beck (1982)
*     equations (20) and (15), (16), (17a)
*     RFLB82(RXIN,RAIN)=H(RXIN*RAIN,1/RAIN)=H(RHX,RHP)=H(X,p)
      IMPLICIT NONE
      DOUBLE PRECISION RXIN, RAIN
      DOUBLE PRECISION RPI, R2OPI
*                      Abramowitz&Stegun p.3
      PARAMETER       (RPI=3.14159265358979323846264 3D0,
     &                 R2OPI=2.D0/RPI)
      DOUBLE PRECISION RHPXLARGE      , RHPXSMALL         , RHXBIG
      PARAMETER       (RHPXLARGE=10.D0, RHPXSMALL=0.0001D0, RHXBIG=4.D0)
      DOUBLE PRECISION RHP1PLUS      , RHP1MINUS,       RASMALL
      PARAMETER       (RHP1PLUS=1.3D0, RHP1MINUS=0.6D0, RASMALL=1D-30)
      DOUBLE PRECISION RX, RA, RHX, RHP, RHPX
      DOUBLE PRECISION RFERFC, RFLBSU, RFBARST
*     work with absolute values because of symmetry relations
      RX  = ABS(RXIN)
      RA  = ABS(RAIN)
      RHX = RA*RX                                     ! X of H(X,p)
      IF (RA.LT. RASMALL) THEN
         RHP = 1.D0/RASMALL
      ELSE
         RHP = 1.D0/RA                                ! p of H(X,p)
      END IF
```

```
      RHPX= RX                                              ! =pX
      IF (RHPX.GT.RHPXLARGE) THEN
*        pX is large
*        this is never reached when called from RFIEER
*        where this condition is taken care of without
*        calling this routine RFLB82
         RFLB82=0.D0
      ELSE IF (RHPX.LT.RHPXSMALL) THEN
*        pX is small
         IF (RHP.GT.1.D0) THEN
*           Litkouhi & Beck eq.21a: H(X,p)=2/pi*(arctan(1/p)-pX
            RFLB82=R2OPI*ATAN(RA)-RHPX
         ELSE
*           Litkouhi & Beck eq.21b:
*           H(X,p)=1-erf(px)erf(x)-2/pi*(arctan(1/p)-pX
            RFLB82=1.D0-(1.D0-RFERFC(RHPX))*(1.D0-RFERFC(RHX))
     &         -R2OPI*ATAN(RA)-RHPX
         END IF
      ELSE
*        pX is neither large nor small
         IF (RHX.GT.RHXBIG) THEN
*           X is large
*           Litkouhi & Beck: equation 20
            RFLB82=RFERFC(RHPX)
         ELSE
*           X is not large
            IF (RHP.GT.RHP1PLUS) THEN
*              p is decisively greater than 1
*              Litkouhi & Beck: equation 16
               RFLB82=R2OPI*EXP(-RX*RX)*RFLBSU(RX,RA)
            ELSE IF (RHP.LT.RHP1MINUS) THEN
*              p is decisively less than 1
*              Litkouhi & Beck: equation 15
               RFLB82=1.D0-(1.D0-RFERFC(RHX))*(1.D0-RFERFC(RX))-
     &            R2OPI*EXP(-RHX*RHX)*RFLBSU(RHX,RHP)
            ELSE IF (RHP.EQ.1.D0) THEN
*              p is exactly 1
*              Litkouhi & Beck: equation 17a
               RFLB82=.5D0-.5D0*(1.D0-RFERFC(RX))**2
            ELSE
*              p is about 1
*              Barnes & Strack ICAEM2003
               RFLB82=RFBARST(RHPX,RHX,RHP)
            END IF
         END IF
      END IF
*     H(X,-p) =-H(X,p)
*     H(-X,p) = H(X,p)
*     H(a(-x),1/a) = H(-ax,1/a) = H(ax,1/a)
*     H(-ax,1/(-a))= H(-ax,-1/a)= H(ax,-1/a) = H(ax,1/a)
      IF (RAIN.LT.0.D0) RFLB82=-RFLB82
```

```
      RETURN
      END
```

The function LBSU which the above function LB82 calls is listed below:

```
      DOUBLE PRECISION FUNCTION RFLBSU(RX,RA)
*                           (-1)**n * e_n(RX**2) * RA**(2n+1)
*     sum for n=0 to inf of ---------------------------------
*                                        2n+1
*     used for formula (15) and (16) of Litkouhi and Beck (1982)
*     e_n from Abramowitz and Stegun: page 262, equation 6.5.11
      IMPLICIT NONE
      DOUBLE PRECISION RX, RA
      INTEGER          NORDMX
      PARAMETER       (NORDMX=100)
      DOUBLE PRECISION RACC      , RCUT
      PARAMETER       (RACC=1.D-8, RCUT=50.D0)
      DOUBLE PRECISION RSIGN, RXNONF, RENRX, RAA, RA2NP1, R2NP1,
     &                 R0, RSUM, RI
      INTEGER          I
      IF (RX.GT.RCUT) THEN
         RFLBSU = 0.D0
      ELSE
         RSIGN = 1.D0
         RXNONF= 1.D0
         RENRX = 1.D0
         RAA   = RA**2
         RA2NP1= RA
         R2NP1 = 1.D0
         RSUM  = RA
         R0    = ABS(RA)*RACC
         DO I=1,NORDMX
            RSIGN = -RSIGN
            RXNONF= RXNONF*RX/DBLE(I)
            RENRX = RENRX+RXNONF
            R2NP1 = R2NP1+2.D0
            RA2NP1= RA2NP1*RAA
            RI    = RSIGN*RENRX*RA2NP1/R2NP1
            RSUM  = RSUM+RI
            IF (ABS(RI).LE.R0) EXIT
         END DO
         RFLBSU=RSUM
      END IF
      RETURN
      END
```

The function BARST which the above function LB82 calls implements an approximation that has been presented by Barnes & Strack (2003). The listing is given below:

```
      DOUBLE PRECISION FUNCTION RFBARST(RHPX,RHX,RHP)
*     approximation for H(X,p) function in the range
*     0.0001 <= pX <= 4, X<=4, and 0.6 <= p <= 1.3
*     the function H(X,p) is defined by Litkouhi and Beck (1982)
*     the approximation was formulated by Barnes & Strack in
*     a paper for the ICAEM conference in France in April 2003
```

```
*     which was cancelled unfortunately
      IMPLICIT NONE
      DOUBLE PRECISION RHPX, RHX, RHP
      DOUBLE PRECISION RPI, R2OPI
*                      Abramowitz&Stegun p.3
      PARAMETER       (RPI=3.14159265358979323846264 3D0,
     &                 R2OPI=2.D0/RPI)
      INTEGER          NMAX
      PARAMETER       (NMAX=100)
      DOUBLE PRECISION RTOL
      PARAMETER       (RTOL=1.D-8)
      INTEGER          I
      DOUBLE PRECISION RPHIA, RPHIB, RPHIC, RSUM, RTEST, RI, RHXX, RHPP,
     &                 R1MPPO2, R1PPPN
      DOUBLE PRECISION RFERFC, RFBRF
      RTEST  = RFBRF(RHX)
      RPHIA  = 0.5D0*RTEST
      RPHIB  = RTEST**2
      RSUM   = RPHIB
      RTEST  = ABS(RPHIB)*RTOL
      RHXX   = RHX**2
      RHPP   = RHP**2
      R1MPPO2=.5D0-.5D0*RHPP
      R1PPPN =1.D0
      DO I=1,NMAX
         RPHIC = (  (DBLE(2*I-1)-2.D0*RHXX)*RPHIB
     &            + 4.D0*RHXX*RPHIA - 0.5D0
     &           ) / DBLE(I)
         R1PPPN= R1PPPN*R1MPPO2
         RI    = R1PPPN*RPHIC
         RSUM  = RSUM+RI
         IF (ABS(RI).LT.RTEST) EXIT
         RPHIA = RPHIB
         RPHIB = RPHIC
      END DO
      RFBARST = RFERFC(RHPX)-R2OPI*RHP*EXP((-1.D0-RHPP)*RHXX)*RSUM
      RETURN
      END
```

The function BRF which the above function BARST calls is listed below:

```
      DOUBLE PRECISION FUNCTION RFBRF(RHX)
*     function from Barkley Rosser (1948): equation 6-2
*     that is used in the approximation for H(X,p) by
*     Barnes & Strack in a paper for the ICAEM conference
*     in France in April 2003 which was cancelled unfortunately
      IMPLICIT NONE
      DOUBLE PRECISION RHX
      DOUBLE PRECISION RSQRPI, RSQRPIO2
*                      Abramowitz & Stegun: page 3
      PARAMETER       (RSQRPI=1.77245385090551602729816 7D0,
     &                 RSQRPIO2=RSQRPI/2.D0)
      DOUBLE PRECISION RFERFC
```

```
      RFBRF = EXP(RHX**2)*RSQRPIO2*RFERFC(RHX)
      RETURN
      END
```

The function ERFC which is used in a number of the previously mentioned functions is taken from Abramowitz & Stegun (1972)

```
*     complementary error function erfc(x)= 1-erf(x)
*     using formula 7.1.26 on page 299 of Abramowitz & Stegun
      DOUBLE PRECISION FUNCTION RFERFC(RX)
      IMPLICIT NONE
      DOUBLE PRECISION RX
      DOUBLE PRECISION RP, RA1, RA2, RA3, RA4, RA5
      PARAMETER       (RP=    .3275911D0   , RA1= .254829592D0,
     &                 RA2= -.284496736D0, RA3=1.421413741D0,
     &                 RA4=-1.453152027D0, RA5=1.061405429D0)
*     the function returns practically zero for values
*     of RX greater than 10. ( RFERFC(9.)=4.28E-37 )
      DOUBLE PRECISION RXABS, RT
      RXABS=ABS(RX)
      RT=1.D0/(1.D0+RP*RXABS)
      RFERFC=RT*(RA1+RT*(RA2+RT*(RA3+RT*(RA4+RT*RA5))))*EXP(-RX*RX)
      IF (RX .LT. 0.D0) RFERFC=2.D0-RFERFC
      RETURN
      END
```

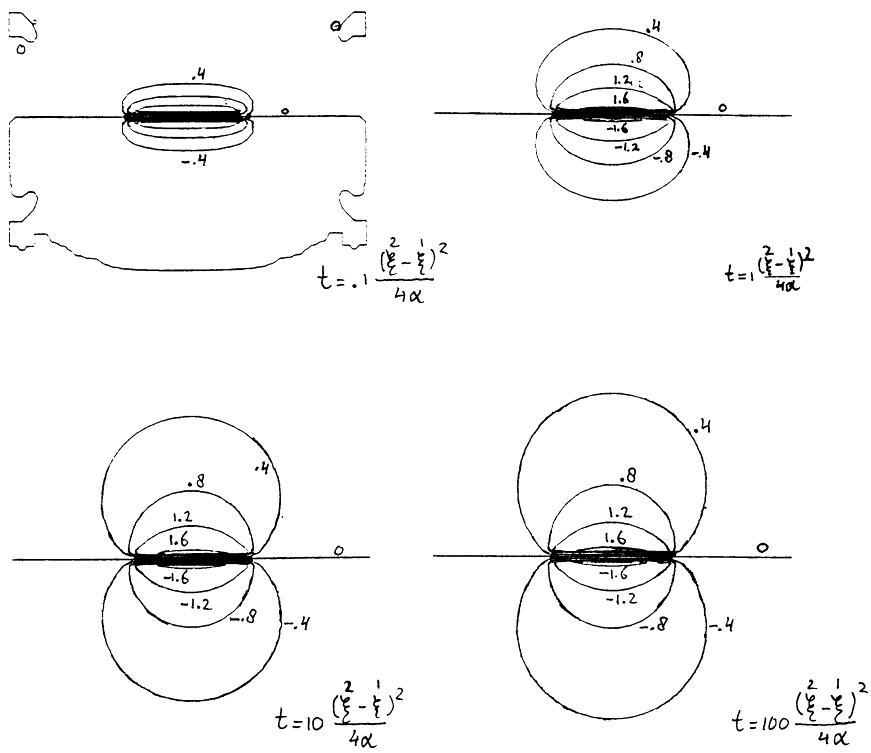

figure A.1. *Contours at different times*

Contour plots in the ξ,η plane at different times obtained with this approximation are figure A.1. The plots illustrate the behavior of the function. When the time t is equal to zero the value of the function is zero throughout the entire plane. For positive times there is a constant discontinuity along the ξ-axis between ξ_1 and ξ_2. The value at infinity remains zero, while the function becomes constant for large values of the time.

The definition of the complementary error function in the integrand of the function (A.1) is (4.39)

$$\mathrm{erfc}x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du \tag{A.2}$$

Using the equations (7.1.2), (7.1.19) and (B.4.8) in Abramowitz and Stegun (1972), its derivative is found to be

$$\frac{d}{dx}\mathrm{erfc}x = -\frac{2}{\sqrt{\pi}}e^{-x^2} \tag{A.3}$$

The derivative of the function (A.1) with respect to ξ is equal to

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^\infty e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta})-\mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du\\
&=\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^\infty\Big\{\frac{\partial}{\partial\xi}\big[e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta})-\mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\big]\Big\}\,du\\
&=\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^\infty e^{-u^2}\{-\frac{2}{\sqrt{\pi}}e^{-u^2\frac{(\xi-\xi_2)^2}{\eta^2}}\frac{u}{\eta}+\frac{2}{\sqrt{\pi}}e^{-u^2\frac{(\xi-\xi_1)^2}{\eta^2}}\frac{u}{\eta}\}\,du\\
&=\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^\infty \frac{1}{\sqrt{\pi}}\{\frac{-2u}{\eta}e^{-u^2\frac{(\xi-\xi_2)^2+\eta^2}{\eta^2}}-\frac{-2u}{\eta}e^{-u^2\frac{(\xi-\xi_1)^2+\eta^2}{\eta^2}}\}\,du\\
&=\frac{1}{\sqrt{\pi}}\Big[\frac{\eta}{(\xi-\xi_2)^2+\eta^2}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^\infty\frac{(\xi-\xi_2)^2+\eta^2}{\eta}\frac{-2u}{\eta}e^{-u^2\frac{(\xi-\xi_2)^2+\eta^2}{\eta^2}}\,du\\
&\qquad-\frac{\eta}{(\xi-\xi_1)^2+\eta^2}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^\infty\frac{(\xi-\xi_1)^2+\eta^2}{\eta}\frac{-2u}{\eta}e^{-u^2\frac{(\xi-\xi_1)^2+\eta^2}{\eta^2}}\,du\Big]\\
&=\frac{1}{\sqrt{\pi}}\Big[\frac{\eta}{(\xi-\xi_2)^2+\eta^2}e^{-u^2\frac{(\xi-\xi_2)^2+\eta^2}{\eta^2}}\Big|_{\sqrt{\frac{\eta^2}{4\alpha t}}}^\infty-\frac{\eta}{(\xi-\xi_1)^2+\eta^2}e^{-u^2\frac{(\xi-\xi_1)^2+\eta^2}{\eta^2}}\Big|_{\sqrt{\frac{\eta^2}{4\alpha t}}}^\infty\Big]
\end{aligned} \tag{A.4}$$

so that

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^\infty e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta})-\mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du\\
&=-\frac{\eta}{\sqrt{\pi}}\{\frac{e^{-\frac{(\xi-\xi_2)^2+\eta^2}{4\alpha t}}}{(\xi-\xi_2)^2+\eta^2}-\frac{e^{-\frac{(\xi-\xi_1)^2+\eta^2}{4\alpha t}}}{(\xi-\xi_1)^2+\eta^2}\}
\end{aligned} \tag{A.5}$$

Using Leibnitz' rule for the derivative of an integral (see e.g. Wylie and Barrett (1982) page 454)

$$\frac{\partial}{\partial x}\int_{g(x)}^{h(x)} f(x,u)\,du = \int_{g(x)}^{h(x)}\frac{\partial f(x,u)}{\partial x}\,du + f(x,u)\big|_{u=h(x)}\frac{dh(x)}{dx} - f(x,u)\big|_{u=g(x)}\frac{dg(x)}{dx} \tag{A.6}$$

the derivative with respect to η of the function (A.1) is equal to

$$\begin{aligned}
&\frac{\partial}{\partial\eta}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du \\
&= \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{-\frac{2}{\sqrt{\pi}}e^{-u^2\frac{(\xi-\xi_2)^2}{\eta^2}}(-u\frac{\xi-\xi_2}{\eta^2}) + \frac{2}{\sqrt{\pi}}e^{-u^2\frac{(\xi-\xi_1)^2}{\eta^2}}(-u\frac{\xi-\xi_1}{\eta^2})\}\,du \\
&\quad +0 \\
&\quad - \left[e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\Big|_{u=\sqrt{\frac{\eta^2}{4\alpha t}}}\right]\frac{1}{2\sqrt{\alpha t}}
\end{aligned} \tag{A.7}$$

for which the following expression is obtained after an integration similar to the one in (A.4)

$$\begin{aligned}
&\frac{\partial}{\partial\eta}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du \\
&= \frac{1}{\sqrt{\pi}}\{\frac{\xi-\xi_2}{(\xi-\xi_2)^2+\eta^2}e^{-\frac{(\xi-\xi_2)^2+\eta^2}{4\alpha t}} - \frac{\xi-\xi_1}{(\xi-\xi_1)^2+\eta^2}e^{-\frac{(\xi-\xi_1)^2+\eta^2}{4\alpha t}}\} \\
&\quad - \frac{e^{-\frac{\eta^2}{4\alpha t}}}{2\sqrt{\alpha t}}\{\mathrm{erfc}(\frac{\xi-\xi_2}{2\sqrt{\alpha t}}) - \mathrm{erfc}(\frac{\xi-\xi_1}{2\sqrt{\alpha t}})\}
\end{aligned} \tag{A.8}$$

Both derivatives (A.5) and (A.8) do not exist at the straight line-segment from $(\xi_1, 0)$ to $(\xi_2, 0)$. However the limits for η appoaching zero from above and below are equal for both derivatives

$$\begin{aligned}
&\lim_{\eta\downarrow 0}\frac{\partial}{\partial\xi}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du \\
&\quad \lim_{\eta\uparrow 0}\frac{\partial}{\partial\xi}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
&\lim_{\eta\downarrow 0}\frac{\partial}{\partial\eta}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du \\
&\quad \lim_{\eta\uparrow 0}\frac{\partial}{\partial\eta}\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du
\end{aligned} \tag{A.10}$$

The function (A.1) fulfills the differential equation (2.21) in the entire plane except the line-segment from $(\xi_1, 0)$ to $(\xi_2, 0)$.

$$\{\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} - \frac{1}{\alpha}\frac{\partial}{\partial t}\}\{\int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du\} = 0 \tag{A.11}$$

A.1 singularity.

The function has a discontinuity along the ξ-axis between ξ_1 and ξ_2 of size $2\sqrt{\pi}$. It is shown below that this follows from properties of the complementary error function

$$\mathrm{erfc}(-x) = 2 - \mathrm{erfc}(x) \tag{A.12}$$

$$\lim_{x\to\infty}\mathrm{erfc}(x) = 0 \tag{A.13}$$

so that

$$\lim_{x\to-\infty}\mathrm{erfc}(x) = 2 \tag{A.14}$$

If η is equal to zero the variable of integration u does not attain negative values, so that the signs of the arguments of the complementary error functions in the integrand are determined by ξ, η and ξ_1, ξ_2. On the ξ-axis the difference of the error functions is equal to

$$\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta}) = \begin{cases} 2-2=0 & \eta\downarrow 0, \xi<\xi_1<\xi_2 \\ 2-0=+2 & \eta\downarrow 0, \xi_1<\xi<\xi_2 \\ 0-0=0 & \eta\downarrow 0, \xi_1<\xi_2<\xi \\ 0-0=0 & \eta\uparrow 0, \xi<\xi_1<\xi_2 \\ 0-2=-2 & \eta\uparrow 0, \xi_1<\xi<\xi_2 \\ 2-2=0 & \eta\uparrow 0, \xi_1<\xi_2<\xi \end{cases} \tag{A.15}$$

The integrand in (A.1) then is equal to zero on the ξ-axis except in between ξ_1 and ξ_2

$$e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\} = \begin{cases} +2e^{-u^2} & \eta\downarrow 0, \xi_1<\xi<\xi_2 \\ -2e^{-u^2} & \eta\uparrow 0, \xi_1<\xi<\xi_2 \\ 0 & \eta=0, \xi<\xi_1 \vee \xi>\xi_2 \end{cases} \tag{A.16}$$

So that the integral has a jump with a magnitude of $2\sqrt{\pi}$ across the ξ-axis between ξ_1 and ξ_2

$$\begin{aligned} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du &= \begin{cases} \int_0^\infty +2e^{-u^2}\,du & \eta\downarrow 0, \xi_1<\xi<\xi_2 \\ \int_0^\infty -2e^{-u^2}\,du & \eta\uparrow 0, \xi_1<\xi<\xi_2 \\ 0 & \eta=0, \xi<\xi_1 \vee \xi>\xi_2 \end{cases} \\ &= \begin{cases} +\sqrt{\pi} & \eta\downarrow 0, \xi_1<\xi<\xi_2 \\ -\sqrt{\pi} & \eta\uparrow 0, \xi_1<\xi<\xi_2 \\ 0 & \eta=0, \xi<\xi_1 \vee \xi>\xi_2 \end{cases} \end{aligned} \tag{A.17}$$

The function is bounded is the entire plane. Therefor the limit for η going to zero both from above and from below for η times the function exists

$$\begin{aligned} \lim_{\eta\downarrow 0}\eta^a \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du = 0 \qquad a>0 \\ \lim_{\eta\uparrow 0}\eta^a \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\xi_2}{\eta}) - \mathrm{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du = 0 \qquad a>0 \end{aligned} \tag{A.18}$$

The limit is equal to zero independent of the direction from which η approaches zero, so that the function times a positive power of η is continuous everywhere.

A.2 infinity. It is shown below that the function is zero at infinity

$$\begin{aligned}
&\lim_{\sqrt{\xi^2+\eta^2}\to\infty} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\operatorname{erfc}(u\frac{\xi-\xi_2}{\eta}) - \operatorname{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du \\
&\simeq (\xi_2-\xi_1) \lim_{\sqrt{\xi^2+\eta^2}\to\infty} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} \frac{e^{-u^2}\{\operatorname{erfc}(u\frac{\xi-\xi_2}{\eta}) - \operatorname{erfc}(u\frac{\xi-\xi_1}{\eta})\}}{\xi_2-\xi_1}\,du \\
&= (\xi_2-\xi_1) \lim_{\substack{\Delta\xi\to 0\\ \sqrt{\xi^2+\eta^2}\to\infty}} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2} \frac{\operatorname{erfc}(u\frac{\xi-\frac{1}{2}(\xi_1+\xi_2)-\frac{1}{2}\Delta\xi}{\eta}) - \operatorname{erfc}(u\frac{\xi-\frac{1}{2}(\xi_1+\xi_2)+\frac{1}{2}\Delta\xi}{\eta})}{\Delta\xi}\,du \\
&= \lim_{\sqrt{\xi^2+\eta^2}\to\infty} -\frac{(\xi_2-\xi_1)}{\sqrt{\pi}} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\frac{\partial}{\partial\xi}\{\operatorname{erfc}(u\frac{\xi-\frac{1}{2}(\xi_1+\xi_2)}{\eta})\}\,du \\
&= \lim_{\sqrt{\xi^2+\eta^2}\to\infty} -\frac{\xi_2-\xi_1}{\sqrt{\pi}} \frac{\eta}{(\xi-\frac{1}{2}(\xi_1+\xi_2))^2+\eta^2} e^{-\frac{(\xi-\frac{1}{2}(\xi_1+\xi_2))^2+\eta^2}{4\alpha t}} \\
&= 0
\end{aligned} \tag{A.19}$$

From (A.19) it follows that the function behaves at infinity like

$$-\frac{\eta}{\xi^2+\eta^2} e^{-\frac{\xi^2+\eta^2}{4\alpha t}} \tag{A.20}$$

so that because of the limit

$$\lim_{x\to\infty} x^a e^{-x} = 0 \tag{A.21}$$

(Abramowitz and Stegun (1972) equation 4.2.20) the integral times $\xi^a\eta^b$ vanishes at infinity also

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty} \xi^a\eta^b \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\operatorname{erfc}(u\frac{\xi-\xi_2}{\eta}) - \operatorname{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du = 0 \tag{A.22}$$

A.3 time zero. The limit of the fucntion for positive times approaching zero is equal to zero. The lower bound of the integral approaches the upperbound, while the integrand remains bounded

$$\lim_{t\downarrow 0} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\operatorname{erfc}(u\frac{\xi-\xi_2}{\eta}) - \operatorname{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du = \lim_{b\to\infty}\int_b^{\infty} e^{-u^2} a\,du = 0 \tag{A.23}$$

A.4 large times. The function becomes steady for large values of time. The discontinuity remains and the function becomes equal to two arctangents

$$\begin{aligned}
&\lim_{t\to\infty} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\operatorname{erfc}(u\frac{\xi-\xi_2}{\eta}) - \operatorname{erfc}(u\frac{\xi-\xi_1}{\eta})\}\,du \\
&= \int_\xi \Big[\lim_{t\to\infty} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} \frac{\partial}{\partial\xi}[e^{-u^2}\{\operatorname{erfc}(u\frac{\xi-\xi_2}{\eta}) - \operatorname{erfc}(u\frac{\xi-\xi_1}{\eta})\}]\,du\Big]\,d\xi \\
&= \int_\xi \left[\lim_{t\to\infty} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} e^{-u^2}\{\frac{-2}{\sqrt{\pi}}\frac{u}{\eta}e^{-\eta^2(\frac{\xi-\xi_2}{\eta})^2} - \frac{-2}{\sqrt{\pi}}\frac{u}{\eta}e^{-\eta^2(\frac{\xi-\xi_1}{\eta})^2}\}\,du\right]\,d\xi \\
&= \int_\xi \lim_{t\to\infty}[-\frac{1}{\sqrt{\pi}\eta}\{\frac{e^{-\frac{(\xi-\xi_2)^2+\eta^2}{4\alpha t}}}{(\frac{\xi-\xi_2}{\eta})^2+1} - \frac{e^{-\frac{(\xi-\xi_2)^2+\eta^2}{4\alpha t}}}{(\frac{\xi-\xi_1}{\eta})^2+1}\}]\,d\xi \\
&= \int_\xi [-\frac{1}{\sqrt{\pi}\eta}\{\frac{1}{(\frac{\xi-\xi_2}{\eta})^2+1} - \frac{1}{(\frac{\xi-\xi_1}{\eta})^2+1}\}]\,d\xi \\
&= -\frac{1}{\sqrt{\pi}}\{\arctan(\frac{\xi-\xi_2}{\eta}) - \arctan(\frac{\xi-\xi_1}{\eta})\} \\
&= \frac{1}{\sqrt{\pi}}\{\arctan(\frac{\eta}{\xi-\xi_2}) - \arctan(\frac{\eta}{\xi-\xi_1})\}
\end{aligned} \tag{A.24}$$

appendix B. limits involving E1

In this appendix limits are given for the function

$$\mathrm{E}_1(\frac{x^2+y^2}{4\alpha t}) \tag{B.1}$$

where the exponential integral E_1 is defined as (4.17)

$$\mathrm{E}_1(x) = \int_x^\infty \frac{e^{-u}}{u}\, du \tag{B.2}$$

The derivative of the exponential integral can be found from the equations (5.1.26) and (5.1.4) in Abramowitz and Stegun (1972)

$$\frac{d}{dx}\mathrm{E}_1(x) = -\mathrm{E}_0(x) = -\int_1^\infty e^{-xu} du \tag{B.3}$$

which gives

$$\frac{d}{dx}\mathrm{E}_1(x) = -\frac{e^{-x}}{x} \tag{B.4}$$

Using (B.4) the derivatives of (B.1) with respect to x and y are found to be equal to

$$\frac{\partial}{\partial x}\mathrm{E}_1(\frac{x^2+y^2}{4\alpha t}) = -2\frac{x}{x^2+y^2}e^{-\frac{x^2+y^2}{4\alpha t}} \tag{B.5}$$

and

$$\frac{\partial}{\partial y}\mathrm{E}_1(\frac{x^2+y^2}{4\alpha t}) = -2\frac{y}{x^2+y^2}e^{-\frac{x^2+y^2}{4\alpha t}} \tag{B.6}$$

singularity

At the point $(0,0)$ the function has a logarithmic singularity as can be seen from an approximation for the exponential integral for small arguments (Abramowitz and Stegun, 1972)

$$\mathrm{E}_1(x) \simeq -\log x - \gamma \qquad x \ll 1 \tag{B.7}$$

where γ is Euler's constant. However multiplied by the distance to the origin the function is zero there

$$\begin{aligned}\lim_{\sqrt{x^2+y^2}\to 0}[(x^2+y^2)^a \mathrm{E}_1(\frac{x^2+y^2}{4\alpha t})] &= \lim_{\sqrt{x^2+y^2}\to 0}[(x^2+y^2)^a\{-\log(x^2+y^2)+\log(4\alpha t)-\gamma\}] \\ &= -\lim_{\sqrt{x^2+y^2}\to 0}[(x^2+y^2)^a \log(x^2+y^2)] \\ &= 0\end{aligned} \tag{B.8}$$

because of the limit

$$\lim_{x\to 0} x^a \log x = 0 \qquad a > 0 \tag{B.9}$$

(Abramowitz and Stegun (1972), equation 4.1.31).

infinity

For the behavior at infinity bounding values are considered from Abramowitz and Stegun (1972), equation 5.1.19

$$\frac{1}{x+1} < e^x \mathrm{E}_1(x) < \frac{1}{x} \qquad x > 0 \tag{B.10}$$

so that

$$\frac{1}{x+1} e^{-x} < \mathrm{E}_1(x) < \frac{1}{x} e^{-x} \qquad x > 0 \tag{B.11}$$

It follows that $\mathrm{E}_1(x)$ goes to zero as $\frac{1}{x}e^{-x}$ for large values of x, so that the limit for the exponential integral times a power of x also goes to zero

$$\lim_{x\to\infty} x^a \mathrm{E}_1(x) = 0 \qquad x > 0 \tag{B.12}$$

so that

$$\lim_{\sqrt{x^2+y^2}\to\infty} x^a y^b \mathrm{E}_1\left(\frac{x^2+y^2}{4\alpha t}\right) = 0 \tag{B.13}$$

time zero

The function is zero at time zero

$$\begin{aligned} \lim_{t\to 0} \mathrm{E}_1\left(\frac{x^2+y^2}{4\alpha t}\right) &= \lim_{b\to\infty} \mathrm{E}_1(b) \\ &= \lim_{b\to\infty} \int_b^\infty \frac{e^{-u}}{u}\, du \\ &= 0 \end{aligned} \tag{B.14}$$

large times

The limit for large values of t can be found by using (B.7) to write

$$\begin{aligned} \lim_{t\to\infty} \mathrm{E}_1\left(\frac{x^2+y^2}{4\alpha t}\right) &= -\log\left(\frac{x^2+y^2}{4\alpha t}\right) - \gamma \\ &= -\log\left(\frac{x^2+y^2}{R^2}\right) + \log\left(\frac{4\alpha t}{R^2}\right) - \gamma \end{aligned} \tag{B.15}$$

where R is a length.

appendix C. limits involving $e^{-\frac{\eta^2}{4\alpha t}}\mathrm{erfc}(\frac{\xi-\xi_j}{2\sqrt{\alpha t}})$

In this appendix limits are given for the function

$$e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\xi_2}{2\sqrt{\alpha t}})-\mathrm{erfc}(\frac{\xi-\xi_1}{2\sqrt{\alpha t}})\} \tag{C.1}$$

The derivative with respect to ξ is equal to

$$\frac{\partial}{\partial\xi}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\xi_2}{2\sqrt{\alpha t}})-\mathrm{erfc}(\frac{\xi-\xi_1}{2\sqrt{\alpha t}})\}=-\frac{1}{\sqrt{\pi}\sqrt{\alpha t}}\{e^{-\frac{(\xi-\xi_2)^2+\eta^2}{4\alpha t}}-e^{-\frac{(\xi-\xi_1)^2+\eta^2}{4\alpha t}}\} \tag{C.2}$$

The derivative with respect to η is equal to

$$\frac{\partial}{\partial\eta}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\xi_2}{2\sqrt{\alpha t}})-\mathrm{erfc}(\frac{\xi-\xi_1}{2\sqrt{\alpha t}})\}=-\frac{\eta}{2\alpha t}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\xi_2}{2\sqrt{\alpha t}})-\mathrm{erfc}(\frac{\xi-\xi_1}{2\sqrt{\alpha t}})\} \tag{C.3}$$

singularity

The function does not have singularities.

infinity

From equation (A.21) it follows that

$$\lim_{\eta\to\infty}\eta^a e^{-\frac{\eta^2}{4\alpha t}}=0 \tag{C.4}$$

Limiting values for the complementary error function can be derived from the expression

$$\frac{1}{x+\sqrt{x^2+2}}<e^{x^2}\int_x^\infty e^{-t^2}\,dt\leq\frac{1}{x+\sqrt{x^2+\frac{4}{\pi}}} \tag{C.5}$$

(Abramowitz and Stegun (1972) equation 7.1.13). Using the definition of the complementary error function (4.39) this can be written as

$$\frac{\frac{\sqrt{\pi}}{2}e^{-x^2}}{x+\sqrt{x^2+2}}<\mathrm{erfc}(x)\leq\frac{\frac{\sqrt{\pi}}{2}e^{-x^2}}{x+\sqrt{x^2+\frac{4}{\pi}}} \tag{C.6}$$

so that the complementary error function of x goes to zero for large values of x like

$$\lim_{x\to\infty}\mathrm{erfc}(x)=\sqrt{\pi}\frac{e^{-x^2}}{4x} \tag{C.7}$$

and

$$\lim_{\xi\to\infty}\xi^a\mathrm{erfc}(\frac{\xi-\xi_j}{2\sqrt{\alpha t}})=0 \tag{C.8}$$

Combining (C.4) and (C.8) gives

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty}\xi^a\eta^b e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\xi_2}{2\sqrt{\alpha t}})-\mathrm{erfc}(\frac{\xi-\xi_1}{2\sqrt{\alpha t}})\}=0 \tag{C.9}$$

time zero

The limit for small times follows from (A.21) and (C.7)

$$\lim_{t\to 0} e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\xi_2}{2\sqrt{\alpha t}})-\mathrm{erfc}(\frac{\xi-\xi_1}{2\sqrt{\alpha t}})\}=0 \tag{C.10}$$

large times

$$\lim_{t\to\infty} e^{-\frac{\eta^2}{4\alpha t}}=\lim_{a\to 0} e^{-a}=1 \tag{C.11}$$

The power series for the complementary error function is equal to

$$\mathrm{erfc}x=1-\frac{2}{\sqrt{\pi}}x+O(x^3) \tag{C.12}$$

so that

$$\begin{aligned}\mathrm{erfc}(\frac{\xi-\xi_2}{2\sqrt{\alpha t}})-\mathrm{erfc}(\frac{\xi-\xi_1}{2\sqrt{\alpha t}})= &\ 1-\frac{2}{\sqrt{\pi}}\frac{\xi-\xi_2}{2\sqrt{\alpha t}}+O\big((\frac{\xi-\xi_2}{2\sqrt{\alpha t}})^3\big)\\ &-1+\frac{2}{\sqrt{\pi}}\frac{\xi-\xi_2}{2\sqrt{\alpha t}}+O\big((\frac{\xi-\xi_2}{2\sqrt{\alpha t}})^3\big)\\ &=\frac{\xi_2-\xi_1}{\sqrt{\pi\alpha t}}+O(t^{-\frac{3}{2}})\end{aligned} \tag{C.13}$$

from which it follows that

$$\lim_{t\to\infty} t^a e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\xi_2}{2\sqrt{\alpha t}})-\mathrm{erfc}(\frac{\xi-\xi_1}{2\sqrt{\alpha t}})\}=\begin{cases}0 & a<\frac{1}{2}\\ \frac{\xi_2-\xi_1}{\sqrt{\pi\alpha}} & a=\frac{1}{2}\\ \infty & a>\frac{1}{2}\end{cases} \tag{C.14}$$

appendix D.
limits for the exponential function

In this appendix limits are given for the function

$$e^{-\frac{(\xi-\xi_j)^2+\eta^2}{4\alpha t}} \tag{D.1}$$

The derivative with respect to ξ is equal to

$$\frac{\partial}{\partial \xi} e^{-\frac{(\xi-\xi_j)^2+\eta^2}{4\alpha t}} = -\frac{\xi-\xi_j}{2\alpha t} e^{-\frac{(\xi-\xi_j)^2+\eta^2}{4\alpha t}} \tag{D.2}$$

The derivative with respect to η is equal to

$$\frac{\partial}{\partial \eta} e^{-\frac{(\xi-\xi_j)^2+\eta^2}{4\alpha t}} = -\frac{\eta}{2\alpha t} e^{-\frac{(\xi-\xi_j)^2+\eta^2}{4\alpha t}} \tag{D.3}$$

singularity

The function does not have discontinuities or singular points

infinity

From the limit (A.21) it follows that the function vanishes at infinity even when multiplied by ξ or η

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty} \xi^a \eta^b e^{-\frac{(\xi-\xi_j)^2+\eta^2}{4\alpha t}} = 0 \tag{D.4}$$

time zero

The function vanishes for small values of the time

$$\begin{aligned} \lim_{t\to 0} e^{-\frac{(\xi-\xi_j)^2+\eta^2}{4\alpha t}} &= \lim_{b\to\infty} e^{-b} \\ &= 0 \end{aligned} \tag{D.5}$$

large times

For large time the function becomes steady

$$\begin{aligned} \lim_{t\to \infty} e^{-\frac{(\xi-\xi_j)^2+\eta^2}{4\alpha t}} &= \lim_{b\to 0} e^{-b} \\ &= 1 \end{aligned} \tag{D.6}$$

appendix E. derivation and checking of line-sink of arbitrary degree

In this appendix the potential for a transient line-sink of order one and arbitrary degree is derived and checked.

E.1 derivation of line-sink of arbitrary degree.

It was shown in chapter 4 that the potential for a line-sink of order one and degree n can be written as the integral (4.40)

$$\underset{lsn}{\Phi} = \int_0^t -\frac{\underset{n}{\sigma}\tau^n\sqrt{\alpha}}{4\sqrt{\pi}\sqrt{t-\tau}} e^{-\frac{\eta^2}{4\alpha(t-\tau)}} \{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\}\, d\tau \tag{E.1}$$

Using the binomial theorem (Abramowitz and Stegun (1972), equation 3.1.1)

$$\tau^n = \{t-(t-\tau)\}^n = \sum_{k=0}^{k=n}\{\binom{n}{k}(-1)^k t^{n-k}(t-\tau)^k\} \tag{E.2}$$

to change the factors τ in (E.1) into factors $(t-\tau)$, this integral can be rewritten as

$$\begin{aligned}
\underset{lsn}{\Phi} &= -\frac{\underset{n}{\sigma}\sqrt{\alpha}}{4\sqrt{\pi}} \int_0^t \frac{\sum_{k=0}^{k=n}\{\binom{n}{k}(-1)^k t^{n-k}(t-\tau)^k\}}{\sqrt{t-\tau}} \\
&\qquad e^{-\frac{\eta^2}{4\alpha(t-\tau)}} \{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\}\, d\tau \\
&= \sum_{k=0}^{k=n}[\underset{n}{\sigma}\frac{\binom{n}{k}(-1)^{k+1}t^{n-k}\sqrt{\alpha}}{4\sqrt{\pi}} \\
&\qquad \int_0^t (t-\tau)^{k-\frac{1}{2}} e^{-\frac{\eta^2}{4\alpha(t-\tau)}} \{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\}\, d\tau]
\end{aligned} \tag{E.3}$$

In order to evaluate the integrals in this equation the following change of variables is applied

$$\begin{aligned}
u &= \sqrt{\frac{\eta^2}{4\alpha(t-\tau)}} \\
t-\tau &= \frac{\eta^2}{4\alpha u^2} \\
d\tau &= \frac{\eta^2}{2\alpha u^3}\, du
\end{aligned} \tag{E.4}$$

With this change of variables the integral in equation $(E.3)$ becomes

$$\begin{aligned}
&\int_0^t (t-\tau)^{k-\frac{1}{2}} e^{-\frac{\eta^2}{4\alpha(t-\tau)}} \{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{2\sqrt{\alpha(t-\tau)}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{2\sqrt{\alpha(t-\tau)}})\}\, d\tau\} \\
&= \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} (\frac{\eta^2}{4\alpha u^2})^{k-\frac{1}{2}} e^{-u^2} \{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\} \frac{\eta^2}{2\alpha u^3}\, du \\
&= 2(\frac{\eta^2}{4\alpha})^{k+\frac{1}{2}} \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} \frac{1}{u^{2k+2}} e^{-u^2} \{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\, du \\
&= 2(\frac{\eta^2}{4\alpha})^{k+\frac{1}{2}} \overset{k}{I}_1 \qquad\qquad k \geq 0
\end{aligned} \tag{E.5}$$

where $\overset{k}{I}_1$ is equal to

$$\overset{k}{I}_1 = \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} \frac{1}{u^{2k+2}} e^{-u^2} \{\mathrm{erfc}(u\frac{\xi - \overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi - \overset{1}{\xi}}{\eta})\}\, du \qquad k \geq 0 \tag{E.6}$$

The integral $\overset{k}{I}_1$ is integrated by parts as follows:

$$\begin{aligned}
\overset{k}{I}_1 &= \int_{\frac{\eta}{2\sqrt{\alpha t}}}^{\infty} f' g\, du = fg\Big|_{u=\sqrt{\frac{\eta^2}{4\alpha t}}}^{u=\infty} - \int_{\sqrt{\frac{\eta^2}{4\alpha t}}}^{\infty} f g'\, du \\
f'(u) &= \frac{1}{u^{2k+2}} \\
f(u) &= \frac{-1}{2k+1}\frac{1}{u^{2k+1}} \\
g(u) &= e^{-u^2}\{\mathrm{erfc}(u\frac{\xi - \overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi - \overset{1}{\xi}}{\eta})\} \\
g'(u) &= e^{-u^2}\{\frac{-2}{\sqrt{\pi}} e^{-(u\frac{\xi-\overset{2}{\xi}}{\eta})^2}\frac{\xi - \overset{2}{\xi}}{\eta} - \frac{-2}{\sqrt{\pi}} e^{-(u\frac{\xi-\overset{1}{\xi}}{\eta})^2}\frac{\xi - \overset{1}{\xi}}{\eta}\} \\
&\quad + (-2u)e^{-u^2}\{\mathrm{erfc}(u\frac{\xi - \overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi - \overset{1}{\xi}}{\eta})\} \\
&= \frac{-2}{\sqrt{\pi}}\{\frac{\xi - \overset{2}{\xi}}{\eta} e^{-u^2\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{\eta^2}} - \frac{\xi - \overset{1}{\xi}}{\eta} e^{-u^2\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{\eta^2}}\} \\
&\quad - 2ue^{-u^2}\{\mathrm{erfc}(u\frac{\xi - \overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi - \overset{1}{\xi}}{\eta})\}
\end{aligned} \tag{E.7}$$

which yields

$$\begin{aligned}
\overset{k}{I}_1 &= \frac{-1}{2k+1}\frac{1}{u^{2k+1}} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi - \overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi - \overset{1}{\xi}}{\eta})\}\Big|_{u=\sqrt{\frac{\eta}{4\alpha t}}}^{u=\infty} \\
&- \int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} \frac{-1}{2k+1}\frac{1}{u^{2k+1}} \\
&\quad [\frac{-2}{\sqrt{\pi}}\{\frac{\xi - \overset{2}{\xi}}{\eta} e^{-u^2\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{\eta^2}} - \frac{\xi - \overset{1}{\xi}}{\eta} e^{-u^2\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{\eta^2}}\} \\
&\quad - 2ue^{-u^2}\{\mathrm{erfc}(u\frac{\xi - \overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi - \overset{1}{\xi}}{\eta})\}]\, du \qquad k \geq 0
\end{aligned} \tag{E.8}$$

so that

$$\begin{aligned}
\overset{k}{I}_1 &= \frac{1}{2k+1}(\sqrt{\frac{4\alpha t}{\eta^2}})^{2k+1} e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi - \overset{2}{\xi}}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}}) - \mathrm{erfc}(\frac{\xi - \overset{1}{\xi}}{\eta}\sqrt{\frac{\eta^2}{4\alpha t}})\} \\
&+ \frac{-1}{2k+1}\frac{2}{\sqrt{\pi}}\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} \frac{1}{u^{2k+1}}\{\frac{\xi - \overset{2}{\xi}}{\eta} e^{-u^2\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{\eta^2}} - \frac{\xi - \overset{1}{\xi}}{\eta} e^{-u^2\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{\eta^2}}\}\, du \\
&+ \frac{-2}{2k+1}\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} \frac{1}{u^{2k}} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi - \overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi - \overset{1}{\xi}}{\eta})\}\, du \qquad k \geq 0
\end{aligned} \tag{E.9}$$

which, for $k > 0$ can be written as

$$
\begin{aligned}
\overset{k}{I}_1 =& \frac{1}{2k+1}\left(\sqrt{\frac{4\alpha t}{\eta^2}}\right)^{2k+1} e^{-\frac{\eta^2}{4\alpha t}}\left\{\operatorname{erfc}\left(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}\right)-\operatorname{erfc}\left(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}\right)\right\} \\
&+\frac{-1}{2k+1}\frac{2}{\sqrt{\pi}}\left\{\frac{\xi-\overset{2}{\xi}}{\eta}\underset{2}{\overset{k}{I}}_2-\frac{\xi-\overset{1}{\xi}}{\eta}\underset{1}{\overset{k}{I}}_2\right\} \\
&+\frac{-2}{2k+1}\overset{k-1}{I}_1 \qquad\qquad k>0
\end{aligned}
\tag{E.10}
$$

where the integrals $\underset{j}{\overset{k}{I}}_2 \quad (j=1,2)$ are equal to

$$
\underset{j}{\overset{k}{I}}_2 = \int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} \frac{1}{u^{2k+1}} e^{-u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}}\,du \qquad j=1,2 \qquad k\geq 0
\tag{E.11}
$$

This integral can be evaluated with integration by parts, assuming $k > 0$

$$
\begin{aligned}
\underset{j}{\overset{k}{I}}_2 &= \int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} f'g\,du = fg\Big|_{u=\sqrt{\frac{\eta}{4\alpha t}}}^{u=\infty} - \int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} fg'\,du \\
f'(u) &= \frac{1}{u^{2k+1}} \\
f(u) &= \frac{-1}{2k}\frac{1}{u^{2k}} \\
g(u) &= e^{-u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}} \\
g'(u) &= \frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}(-2u)e^{-u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}}
\end{aligned}
\tag{E.12}
$$

which yields

$$
\begin{aligned}
\underset{j}{\overset{k}{I}}_2 =& \frac{-1}{2k}\frac{1}{u^{2k}}e^{-u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}}\Big|_{u=\sqrt{\frac{\eta}{4\alpha t}}}^{u=\infty} \\
&-\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty}\frac{-1}{2k}\frac{1}{u^{2k}}\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}(-2u)e^{-u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}}\,du \\
=& \frac{1}{2k}\left(\frac{4\alpha t}{\eta^2}\right)^k e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}} \\
&-\frac{1}{k}\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty}\frac{1}{u^{2k-1}}\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}e^{-u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}}\,du \\
=& \frac{1}{2k}\left(\frac{4\alpha t}{\eta^2}\right)^k e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}} \\
&-\frac{1}{k}\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty}\frac{1}{u^{2k-1}}e^{-u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}}\,du \qquad j=1,2 \quad k>0
\end{aligned}
\tag{E.13}
$$

so that, using the definition (E.11), the integral $\overset{k}{\underset{j}{I}}_2$ can be written as

$$
\begin{aligned}
\overset{k}{\underset{j}{I}}_2 = {} & \frac{1}{2k}\left(\frac{4\alpha t}{\eta^2}\right)^k e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}} \\
& -\frac{1}{k}\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}\overset{k-1}{\underset{j}{I}}_2 \qquad j=1,2 \qquad k>0
\end{aligned}
\tag{E.14}
$$

The expressions (E.10) and (E.14) for the integrals $\overset{k}{I}_1$ and $\overset{k}{\underset{j}{I}}_2$ are not valid for $k=0$, but once the integrals have been determined for $k=0$ these equations can be used as recursion formulas to obtain the integrals for any value of k. The integral of $\overset{0}{I}_1$ can be determined from equation $(E.9)$

$$
\begin{aligned}
\overset{0}{I}_1 = {} & \sqrt{\frac{4\alpha t}{\eta^2}}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})-\mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
& -\frac{2}{\sqrt{\pi}}\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty}\frac{1}{u}\{\frac{\xi-\overset{2}{\xi}}{\eta}e^{-u^2\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{\eta^2}}-\frac{\xi-\overset{1}{\xi}}{\eta}e^{-u^2\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{\eta^2}}\}\,du \\
& -2\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty}e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta})-\mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\,du
\end{aligned}
\tag{E.15}
$$

Using

$$
\begin{aligned}
& \int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty}\frac{1}{u}e^{-u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}}\,du \\
& = \frac{1}{2}\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty}\frac{1}{u^2}\frac{\eta^2}{(\xi-\overset{j}{\xi})^2+\eta^2}e^{-u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}}2u\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}\,du \\
& = \frac{1}{2}\int_{\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}}^{\infty}\frac{1}{\left[u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}\right]}e^{-\left[u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}\right]}\,d\left[u^2\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{\eta^2}\right] \\
& = \frac{1}{2}\int_{\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}}^{\infty}\frac{e^{-w}}{w}dw \\
& = \mathrm{E}_1(w)\Big|_{w=\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}}
\end{aligned}
\tag{E.16}
$$

equation (E.15) can be written as

$$
\begin{aligned}
\overset{0}{I}_1 = {} & \sqrt{\frac{4\alpha t}{\eta^2}}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})-\mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
& -\frac{1}{\sqrt{\pi}}\{\frac{\xi-\overset{2}{\xi}}{\eta}\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t})-\frac{\xi-\overset{1}{\xi}}{\eta}\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
& -2\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty}e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta})-\mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\,du
\end{aligned}
\tag{E.17}
$$

The integral on the last line of equation $(E.17)$ can not be expressed in terms of functions, that are listed in Abramowitz and Stegun (1972). Barkley Rosser (1948) discusses an integral which is closely related. An approximation adapted from Litkouhi and Beck (1982) is given in appendix A, where the integral is discussed.

The integral $\overset{k}{\underset{j}{I}}_2$ (E.11) for $k = 0$ is equal to

$$\overset{0}{\underset{j}{I}}_2 = \int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} \frac{1}{u} e^{-u^2 \frac{(\xi - \overset{j}{\xi})^2 + \eta^2}{\eta^2}}\, du \qquad j = 1, 2 \tag{E.18}$$

using (E.16), this can be written as

$$\overset{0}{\underset{j}{I}}_2 = \frac{1}{2} \mathrm{E}_1\left(\frac{(\xi - \overset{j}{\xi})^2 + \eta^2}{4\alpha t}\right) \qquad j = 1, 2 \tag{E.19}$$

E.2 line-sink of arbitrary degree.

Now the potential for a line-sink of degree n (equation (E.3)) can be written as

$$\underset{lsn}{\Phi} = \frac{\underset{n}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}} \sum_{k=0}^{k=n} [\tbinom{n}{k}(-1)^{k+1} t^{n-k} (\frac{\eta^2}{4\alpha})^{k+\frac{1}{2}} \overset{k}{I}_1] \tag{E.20}$$

where $\overset{k}{I}_1$ can be calculated from the recursion formulas $(E.10)$ and $(E.14)$ and the expressions $(E.17)$ and $(E.19)$.

Two special cases of the general degree line-sink are worked out in chapter 4: a line-sink with a constant strength after time zero (degree zero) and a line-sink with a linearly increasing strength (degree one).

E.3 checking line-sink of arbitrary degree.

The above potential derived for a line-sink of order one and degree n, will be checked against the initial condition and the boundary condition. The final condition does not apply to elements which have a degree higher than zero. The check of the final condition for a line-sink with a degree zero is given after the special case of the line-sink of degree zero has been presented.

E.3.1 initial condition. The initial condition is that the potential (E.20) is equal to zero everywhere when the time is equal to zero. From equation (E.14) and the limit (24.5) it follows that

$$\begin{aligned} \lim_{t\to 0} \overset{k}{\underset{j}{I}}_2 &= \lim_{t\to 0} \left[\frac{1}{2k} (\frac{4\alpha t}{\eta^2})^k e^{-\frac{(\xi - \overset{j}{\xi})^2 + \eta^2}{4\alpha t}} - \frac{1}{k} \frac{(\xi - \overset{j}{\xi})^2 + \eta^2}{\eta^2} \overset{k-1}{\underset{j}{I}}_2 \right] \\ &= \lim_{t\to 0} -\frac{1}{k} \frac{(\xi - \overset{j}{\xi})^2 + \eta^2}{\eta^2} \overset{k-1}{\underset{j}{I}}_2 \qquad j = 1, 2 \qquad k > 0 \end{aligned} \tag{E.21}$$

which leads to the recursive relation

$$\lim_{t\to 0} \overset{k-1}{\underset{j}{I}}_2 = 0 \Rightarrow \lim_{t\to 0} \overset{k}{\underset{j}{I}}_2 = 0 \qquad j = 1, 2 \tag{E.22}$$

Limit (22.14) applied to equation (E.19) gives

$$\begin{aligned} \lim_{t\to 0} \overset{0}{\underset{j}{I}}_2 &= \lim_{t\to 0} \frac{1}{2} \mathrm{E}_1\left(\frac{(\xi - \overset{j}{\xi})^2 - \eta^2}{4\alpha t}\right) \\ &= 0 \qquad j = 1, 2 \end{aligned} \tag{E.23}$$

Combination of this result with the recursion of equation (E.22) makes it possible to say

$$\lim_{t\to 0} \underset{j}{\overset{k}{I}}_2 = 0 \qquad k \geq 0 \quad j = 1,2 \tag{E.24}$$

Application of this together with the limit (23.10) to the expression (E.10) gives

$$\begin{aligned}
\lim_{t\to 0} \overset{k}{I}_1 &= \lim_{t\to 0} \frac{1}{2k+1}\Big(\sqrt{\frac{4\alpha t}{\eta^2}}\Big)^{2k+1} e^{-\frac{\eta^2}{4\alpha t}} \{\mathrm{erfc}(\frac{\xi - \overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi - \overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
&\quad + \frac{-1}{2k+1}\frac{2}{\sqrt{\pi}}\{\frac{\xi - \overset{2}{\xi}}{\eta}\underset{2}{\overset{k}{I}}_2 - \frac{\xi - \overset{1}{\xi}}{\eta}\underset{1}{\overset{k}{I}}_2\} \\
&\quad + \frac{-2}{2k+1} \overset{k-1}{I}_1 \\
&= \frac{-2}{2k+1} \lim_{t\to 0} \overset{k-1}{I}_1
\end{aligned} \tag{E.25}$$

which leads to the recursion formula

$$\lim_{t\to 0} \overset{k-1}{I}_1 = 0 \Rightarrow \lim_{t\to 0} \overset{k}{I}_1 = 0 \qquad k > 0 \tag{E.26}$$

Equation (E.17) combined with the limits (23.10), (22.14) and (21.23) gives

$$\begin{aligned}
\lim_{t\to 0} \overset{0}{I}_1 &= \lim_{t\to 0} \sqrt{\frac{4\alpha t}{\eta^2}} e^{-\frac{\eta^2}{4\alpha t}} \{\mathrm{erfc}(\frac{\xi - \overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi - \overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
&\quad - \frac{1}{\sqrt{\pi}}\{\frac{\xi - \overset{2}{\xi}}{\eta}\mathrm{E}_1(\frac{(\xi - \overset{2}{\xi})^2 + \eta^2}{4\alpha t}) - \frac{\xi - \overset{1}{\xi}}{\eta}\mathrm{E}_1(\frac{(\xi - \overset{1}{\xi})^2 + \eta^2}{4\alpha t})\} \\
&\quad - 2\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi - \overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi - \overset{1}{\xi}}{\eta})\}\, du \\
&= 0
\end{aligned} \tag{E.27}$$

so that with the recursive relation (E.26)

$$\lim_{t\to 0} \overset{k}{I}_1 = 0 \qquad k \geq 0 \tag{E.28}$$

Substituting this result into the formula for the potential (E.20) shows that the initial condition is satisfied

$$\begin{aligned}
\lim_{t\to 0} \underset{lsn}{\Phi} &= \lim_{t\to 0} \frac{\underset{n}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}} \sum_{k=0}^{k=n} [\tbinom{n}{k}(-1)^{k+1} t^{n-k} (\frac{\eta^2}{4\alpha})^{k+\frac{1}{2}} \overset{k}{I}_1] \\
&= \frac{\underset{n}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}} \sum_{k=0}^{k=n} [0] \\
&= 0
\end{aligned} \tag{E.29}$$

E.3.2 boundary condition. The boundary condition along a line-sink requires that the element removes an amount of water from the aquifer per unit length that is equal to the strength σ. The boundary condition for a line-sink of degree n with strength $\sigma = \underset{n}{\sigma} t^n$ can be stated as the condition that the normal component of the discharge has a jump equal to the strength across the element

$$\lim_{\eta\downarrow 0} Q_\eta - \lim_{\eta\uparrow 0} Q_\eta = -\underset{n}{\sigma} t^n \qquad \overset{1}{\xi} < \xi < \overset{2}{\xi} \tag{E.30}$$

which implies that the derivative with respect to η of the potential (E.20) is discontinuous across the element

$$\begin{cases} \lim_{\eta\downarrow 0} \frac{\partial}{\partial\eta} \underset{lsn}{\Phi} = \frac{1}{2}\underset{n}{\sigma} t^n \\ \lim_{\eta\uparrow 0} \frac{\partial}{\partial\eta} \underset{lsn}{\Phi} = -\frac{1}{2}\underset{n}{\sigma} t^n \end{cases} \qquad \overset{1}{\xi} < \xi < \overset{2}{\xi} \tag{E.31}$$

The derivative of the potential (E.20) with respect to η is equal to

$$\frac{\partial}{\partial\eta}\underset{lsn}{\Phi} = \frac{\underset{n}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}} \sum_{k=0}^{k=n} [\binom{n}{k} \frac{(-1)^{k+1} t^{n-k}}{(4\alpha)^{k+\frac{1}{2}}} \frac{\partial(\eta^{2k+1}\overset{k}{I}_1)}{\partial\eta}] \tag{E.32}$$

Below the condition (E.31) is checked first for $n = 0$ and next for arbitrary degree n. This partial derivative $\frac{\partial(\eta^{2k+1}\overset{k}{I}_1)}{\partial\eta}$ for $k = 0$ is equal to

$$\frac{\partial(\eta\overset{0}{I}_1)}{\partial\eta} = \overset{0}{I}_1 + \eta\frac{\partial\overset{0}{I}_1}{\partial\eta} \tag{E.33}$$

The integral $\overset{0}{I}_1$ is given in (E.17) and its derivative is equal to

$$\begin{aligned}
\frac{\partial\overset{0}{I}_1}{\partial\eta} =& 2\sqrt{\alpha t}\Big[-\frac{1}{\eta^2}e^{-\frac{\eta^2}{4\alpha t}} - \frac{1}{\eta}\frac{\eta}{2\alpha t}e^{-\frac{\eta^2}{4\alpha t}}\Big]\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
& - \frac{1}{\sqrt{\pi}}\Big[-\frac{\xi-\overset{2}{\xi}}{\eta^2}\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) + \frac{\xi-\overset{1}{\xi}}{\eta^2}\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}) \\
& \quad + \frac{\xi-\overset{2}{\xi}}{\eta}\frac{-2\eta}{(\xi-\overset{2}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}} - \frac{\xi-\overset{1}{\xi}}{\eta}\frac{-2\eta}{(\xi-\overset{1}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}\Big] \\
& - 2\Big[\frac{1}{\sqrt{\pi}}\{\frac{\xi-\overset{2}{\xi}}{(\xi-\overset{2}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}} - \frac{\xi-\overset{1}{\xi}}{(\xi-\overset{1}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t}}\} \\
& \quad - \frac{e^{-\frac{\eta^2}{4\alpha t}}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}\Big] \\
=& -2\sqrt{\alpha t}\frac{1}{\eta^2}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
& + \frac{1}{\sqrt{\pi}}\Big[\frac{\xi-\overset{2}{\xi}}{\eta^2}\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) - \frac{\xi-\overset{1}{\xi}}{\eta^2}\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})
\end{aligned} \tag{E.34}$$

where (22.6) and (21.8) have been used. Substitution of (E.17) and (E.34) into (E.33) gives

$$
\begin{aligned}
\frac{\partial(\eta \overset{0}{I}_1)}{\partial\eta} =& \overset{0}{I}_1 + \eta\frac{\partial \overset{0}{I}_1}{\partial\eta} \\
=& \sqrt{\frac{4\alpha t}{\eta^2}} e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
& - \frac{1}{\sqrt{\pi}}\{\frac{\xi-\overset{2}{\xi}}{\eta}\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) - \frac{\xi-\overset{1}{\xi}}{\eta}\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\} \\
& - 2\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\,du \\
& + \eta\Big[-2\sqrt{\alpha t}\frac{1}{\eta^2}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
& \quad + \frac{1}{\sqrt{\pi}}\Big[\frac{\xi-\overset{2}{\xi}}{\eta^2}\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t}) - \frac{\xi-\overset{1}{\xi}}{\eta^2}\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\Big]\Big] \\
=& -2\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty} e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta}) - \mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\,du
\end{aligned}
\tag{E.35}
$$

Using (21.17) the limits for η approaching zero are found to be equal to

$$
\begin{cases}
\lim_{\eta\downarrow 0} \frac{\partial(\eta \overset{0}{I}_1)}{\partial\eta} = -2(+\sqrt{\pi}) = -2\sqrt{\pi} \\
\lim_{\eta\uparrow 0} \frac{\partial(\eta \overset{0}{I}_1)}{\partial\eta} = -2(-\sqrt{\pi}) = 2\sqrt{\pi}
\end{cases}
\tag{E.36}
$$

Substitution of (E.35) into (E.32) for $n = 0$ gives

$$
\begin{cases}
\lim_{\eta\downarrow 0} \frac{\partial}{\partial\eta}\underset{ls0}{\Phi} = \frac{\overset{0}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}}\frac{-1}{\sqrt{4\alpha}}(-2\sqrt{\pi}) = \frac{\overset{0}{\sigma}}{2} \\
\lim_{\eta\uparrow 0} \frac{\partial}{\partial\eta}\underset{ls0}{\Phi} = \frac{\overset{0}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}}\frac{-1}{\sqrt{4\alpha}}2\sqrt{\pi} = -\frac{\overset{0}{\sigma}}{2}
\end{cases}
\tag{E.37}
$$

so that the boundary condition (E.31) is satisfied for $n = 0$.

Now that the boundary condition has been checked for $n = 0$, the boundary condition for arbitrary n will be examined. All partial derivatives $\frac{\partial(\eta^{2k+1}\overset{k}{I}_1)}{\partial\eta}$ in equation (E.32) for $k > 0$ vanish as will be shown below, so that only the one for $k = 0$ contributes to the partial derivative for $\eta = 0$. The partial derivative, which contains the integral $\overset{k}{I}_1$ (E.10), is equal to

$$
\begin{aligned}
\frac{\partial(\eta^{2k+1}\overset{k}{I}_1)}{\partial\eta} =& \frac{\partial}{\partial\eta}\Big[\frac{(2\sqrt{\alpha t})^{2k+1}}{2k+1}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\
& - \frac{1}{2k+1}\frac{2}{\sqrt{\pi}}\eta^{2k}\{(\xi-\overset{2}{\xi})\overset{k}{\underset{2}{I}}_2 - (\xi-\overset{1}{\xi})\overset{k}{\underset{1}{I}}_2\} \\
& - \frac{2}{2k+1}\eta^{2k+1}\overset{k-1}{I}_1\Big] \qquad k > 0
\end{aligned}
\tag{E.38}
$$

which, in turn, is equal to

$$\begin{aligned}\frac{\partial(\eta^{2k+1}\overset{k}{I}_1)}{\partial\eta} = &-\frac{(2\sqrt{\alpha t})^{2k-1}}{2k+1}2\eta e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\ &- \frac{1}{2k+1}\frac{2}{\sqrt{\pi}}\{(\xi-\overset{2}{\xi})\frac{\partial(\eta^{2k}\overset{k}{\underset{2}{I}}_2)}{\partial\eta} - (\xi-\overset{1}{\xi})\frac{\partial(\eta^{2k}\overset{k}{\underset{1}{I}}_2)}{\partial\eta}\} \\ &- \frac{2}{2k+1}\frac{\partial(\eta^{2k+1}\overset{k-1}{I}_1)}{\partial\eta} \qquad k>0 \end{aligned} \tag{E.39}$$

The function $e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})-\mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}$ which is discussed in appendix C, is bounded for $\eta=0$, so that η times this function is equal to zero for $\eta=0$.

$$\lim_{\eta\to 0}\eta e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} = 0 \tag{E.40}$$

so that the term on the first line of equation (E.39) is equal to zero in the limit for η approaching zero. To evaluate the contributions of the terms on the second line of (E.39) the following derivative is evaluated using the expression for $\overset{k}{\underset{j}{I}}_2$ (E.14)

$$\frac{\partial(\eta^{2k}\overset{k}{\underset{j}{I}}_2)}{\partial\eta} = \frac{\partial}{\partial\eta}\Big[\frac{(4\alpha t)^k}{2k}e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}} - \frac{1}{k}\{(\xi-\overset{j}{\xi})^2\eta^{2k-2}+\eta^{2k}\}\overset{k-1}{\underset{j}{I}}_2\Big] \qquad k>0 \tag{E.41}$$

so that

$$\begin{aligned}\frac{\partial(\eta^{2k}\overset{k}{\underset{j}{I}}_2)}{\partial\eta} = &-\frac{(4\alpha t)^{k-1}}{k}\eta e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}} \\ &- \frac{1}{k}(\xi-\overset{j}{\xi})^2\frac{\partial(\eta^{2(k-1)}\overset{k-1}{\underset{j}{I}}_2)}{\partial\eta} - \frac{1}{k}\frac{\partial(\eta^{2k}\overset{k-1}{\underset{j}{I}}_2)}{\partial\eta} \qquad k>0\end{aligned} \tag{E.42}$$

Next it will be shown that this partial derivative is equal to zero.

If the derivative $\frac{\partial(\eta^{2k}\overset{k}{\underset{j}{I}}_2)}{\partial\eta}$ is equal to zero in the limit for $\eta\to 0$ then $\eta^{2k}\overset{k}{\underset{j}{I}}_2$ must behave as

$$\lim_{\eta\to 0}\eta^{2k}\overset{k}{\underset{j}{I}}_2 = a\eta^b \tag{E.43}$$

where a is independent of η and either $b\geq 1$ or $b=0$. The derivative with respect to η would be unbounded at $\eta=0$ if b had a negative value or a value between zero and one. It follows from (E.43) that

$$\lim_{\eta\to 0}\eta^{2m}\overset{k}{\underset{j}{I}}_2 = a\eta^b\eta^{m-k} \tag{E.44}$$

so that for $m\geq k$

$$\lim_{\eta\to 0}\frac{\partial(\eta^{2k}\overset{k}{\underset{j}{I}}_2)}{\partial\eta} = 0 \Rightarrow \lim_{\eta\to 0}\frac{\partial(\eta^{2m}\overset{k}{\underset{j}{I}}_2)}{\partial\eta} = 0 \qquad m\geq k \tag{E.45}$$

The derivative $\frac{\partial(\eta^{2k}\overset{k}{\underset{j}{I}}_2)}{\partial\eta}$ for $k=0$ is determined by using the expression for the integral $\overset{0}{\underset{j}{I}}_2$ (E.19) and equation (22.6)

$$\begin{aligned}\frac{\partial\overset{0}{\underset{j}{I}}_2}{\partial\eta} &= \frac{\partial}{\partial\eta}\frac{1}{2}\mathrm{E}_1\Big(\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}\Big)\\ &= -\frac{\eta}{(\xi-\overset{j}{\xi})^2+\eta^2}e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}}\end{aligned} \tag{E.46}$$

Since the e-power in (E.46) is bounded for $\eta=0$ the following limit is equal to zero

$$\lim_{\eta\to 0}\eta e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}} = 0 \tag{E.47}$$

so that

$$\lim_{\eta\to 0}\frac{\partial\overset{0}{\underset{j}{I}}_2}{\partial\eta} = 0 \tag{E.48}$$

It follows from (E.45) and (E.48) that

$$\lim_{\eta\to 0}\frac{\partial(\eta^2\overset{0}{\underset{j}{I}}_2)}{\partial\eta} = 0 \tag{E.49}$$

With (E.47), (E.48) and (E.49) it follows from (E.42) for $k=1$ that

$$\lim_{\eta\to 0}\frac{\partial(\eta^2\overset{1}{\underset{j}{I}}_2)}{\partial\eta} = 0 \tag{E.50}$$

Alternate application of (E.47) and (E.42) then makes it possible to extend this to

$$\lim_{\eta\to 0}\frac{\partial(\eta^{2k}\overset{k}{\underset{j}{I}}_2)}{\partial\eta} = 0 \qquad k\geq 0 \tag{E.51}$$

To evaluate the term on the third line of (E.39) a relation similar to (E.45) is derived that contains $\overset{k}{I}_1$. If the derivatives $\frac{\partial(\eta^{2k+1}\overset{k}{I}_1)}{\partial\eta}$ are finite in the limits $\eta\downarrow 0$ and $\eta\uparrow 0$ then $\eta^{2k+1}\overset{k}{I}_1$ must behave like

$$\begin{cases}\lim_{\eta\downarrow 0}\eta^{2k+1}\overset{k}{I}_1 = a^+\eta^{b^+}\\ \lim_{\eta\uparrow 0}\eta^{2k+1}\overset{k}{I}_1 = a^-\eta^{b^-}\end{cases} \tag{E.52}$$

where a^+ and a^- are independent of η and b^+ and b^- either equal to zero or greater than or equal to one. It follows from (E.52) that

$$\begin{cases}\lim_{\eta\downarrow 0}\eta^{2m+1}\overset{k}{I}_1 = a^+\eta^{b^++m-k}\\ \lim_{\eta\uparrow 0}\eta^{2m+1}\overset{k}{I}_1 = a^-\eta^{b^-+m-k}\end{cases} \tag{E.53}$$

so that for $m \geq k$

$$\begin{cases} \lim_{\eta\downarrow 0} \frac{\partial(\eta^{2k+1}\overset{k}{I}_1)}{\partial\eta} < \infty \\ \lim_{\eta\uparrow 0} \frac{\partial(\eta^{2k+1}\overset{k}{I}_1)}{\partial\eta} < \infty \end{cases} \Rightarrow \lim_{\eta\to 0} \frac{\partial(\eta^{2m+1}\overset{k}{I}_1)}{\partial\eta} = 0 \qquad m \geq k \tag{E.54}$$

Since $\frac{\partial(\eta\overset{0}{I}_1)}{\partial\eta}$ is finite in the limits $\eta \downarrow 0$ and $\eta \uparrow 0$ (see equation (E.36)) it follows that

$$\lim_{\eta\to 0} \frac{\partial(\eta^3\overset{0}{I}_1)}{\partial\eta} = 0 \tag{E.55}$$

so that the term on the third line of (E.39) also is equal to zero for $k = 1$ in the limit $\eta \to 0$, while the terms on the first and the second line are equal to zero for any k (see (E.40) and (E.51)). Repeated use of the equation (E.40), (E.51) and (E.39) alternating with (E.54) makes it possible to generalize (E.55) to

$$\lim_{\eta\to 0} \frac{\partial(\eta^{2k+1}\overset{k-1}{I}_1)}{\partial\eta} = 0 \qquad k > 0 \tag{E.56}$$

Because of (E.56) all terms in (E.32), the derivative of the potential, vanish for $k > 0$ in the limits for η approaching zero both from above and below. The only term that is not equal to zero for $\eta = 0$, is the term for $k = 0$ of the sum in (E.32), so that the limits become using (E.35) and (21.17)

$$\begin{cases} \lim_{\eta\downarrow 0} \frac{\partial}{\partial\eta} \underset{lsn}{\Phi} = \frac{\underset{n}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}}[\binom{n}{0}\frac{-t^n}{\sqrt{4\alpha}}(-2)\sqrt{\pi}] = \frac{1}{2}\underset{n}{\sigma}t^n \\ \lim_{\eta\uparrow 0} \frac{\partial}{\partial\eta} \underset{lsn}{\Phi} = \frac{\underset{n}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}}[\binom{n}{0}\frac{-t^n}{\sqrt{4\alpha}}(-2)(-\sqrt{\pi})] = -\frac{1}{2}\underset{n}{\sigma}t^n \end{cases} \tag{E.57}$$

which conforms to the boundary condition (E.31).

Thus the potential for the line-sink of arbitrary degree n (E.20) fulfills the boundary condition at the element.

The condition at infinity requires that the potential is equal to zero at infinity

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty} \underset{lsn}{\Phi} = 0 \tag{E.58}$$

or with (E.20)

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty} \frac{\underset{n}{\sigma}\sqrt{\alpha}}{2\sqrt{\pi}} \sum_{k=0}^{k=n} [\binom{n}{k} \frac{(-1)^{k+1}t^{n-k}}{(4\alpha)^{k+\frac{1}{2}}} \eta^{2k+1}\overset{k}{I}_1] = 0 \tag{E.59}$$

The parts that are functions of ξ and η can be rewritten using (E.10)

$$\begin{aligned} \eta^{2k+1}\overset{k}{I}_1 = {} & \frac{(2\sqrt{\alpha t})^{2k+1}}{2k+1} e^{-\frac{\eta^2}{4\alpha t}} \{\mathrm{erfc}(\frac{\xi - \overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}}) - \mathrm{erfc}(\frac{\xi - \overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\} \\ & - \frac{1}{2k+1}\frac{2}{\sqrt{\pi}}\eta^{2k}\{(\xi - \overset{2}{\xi})\underset{2}{\overset{k}{I}}_2 - (\xi - \overset{1}{\xi})\underset{1}{\overset{k}{I}}_2\} \\ & - \frac{2}{2k+1}\eta^{2k+1}\overset{k-1}{I}_1 \qquad k > 0 \end{aligned} \tag{E.60}$$

The expression $\eta^{2m+1}\overset{k}{I}_1$, where m is greater than or equal to k, will be examined rather than $\eta^{2k+1}\overset{k}{I}_1$, because the exponent in the last term in equation (E.60) is $2k+1$ while the superscript is equal $k-1$.

$$\begin{aligned}\eta^{2m+1}\overset{k}{I}_1 = {} & \frac{(2\sqrt{\alpha t})^{2k+1}}{2k+1}\eta^{2(m-k)}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})-\mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}\\ & -\frac{1}{2k+1}\frac{2}{\sqrt{\pi}}\eta^{2m}\{(\xi-\overset{2}{\xi})\overset{k}{\underset{2}{I}}_2-(\xi-\overset{1}{\xi})\overset{k}{\underset{1}{I}}_2\}\\ & -\frac{2}{2k+1}\eta^{2m+1}\overset{k-1}{I}_1 \qquad\qquad m>k \qquad k>0\end{aligned} \tag{E.61}$$

Equation (E.14) is used in order to evaluate the term on the second line

$$\begin{aligned}\eta^{2m}(\xi-\overset{j}{\xi})\overset{k}{\underset{j}{I}}_2 = {} & \frac{(4\alpha t)^k}{2k}(\xi-\overset{j}{\xi})\eta^{2(m-k)}e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}}\\ & -\frac{1}{k}\{(\xi-\overset{j}{\xi})^3\eta^{2m-2}+(\xi-\overset{j}{\xi})\eta^{2m}\}\overset{k-1}{\underset{j}{I}}_2 \qquad j=1,2 \qquad m\geq k \qquad k>0\end{aligned} \tag{E.62}$$

The term for $k=1$ is equal to (see (E.19))

$$\begin{aligned}\eta^{2m}(\xi-\overset{j}{\xi})\overset{1}{\underset{j}{I}}_2 = {} & \frac{4\alpha t}{2}(\xi-\overset{j}{\xi})\eta^{2m-2}e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}}-\{(\xi-\overset{j}{\xi})^3\eta^{2m-2}+(\xi-\overset{j}{\xi})\eta^{2m}\}\overset{0}{\underset{j}{I}}_2\\ = {} & \frac{4\alpha t}{2}(\xi-\overset{j}{\xi})\eta^{2m-2}e^{-\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t}}\\ & -\{(\xi-\overset{j}{\xi})^3\eta^{2m-2}+(\xi-\overset{j}{\xi})\eta^{2m}\}\frac{1}{2}\mathrm{E}_1(\frac{(\xi-\overset{j}{\xi})^2+\eta^2}{4\alpha t})\\ & \qquad j=1,2 \qquad m\geq 0\end{aligned} \tag{E.63}$$

which is equal to zero at infinity

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty}\eta^{2m}(\xi-\overset{j}{\xi})\overset{1}{\underset{j}{I}}_2=0 \qquad m\geq 1 \tag{E.64}$$

because of the limits (23.9) and (22.13). The equation (E.62) can be used repeatedly in this way to get

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty}\eta^{2m}(\xi-\overset{j}{\xi})\overset{k}{\underset{j}{I}}_2=0 \qquad m\geq k \qquad k\geq 0 \tag{E.65}$$

The term for $k=0$ which was excluded in (E.61) is found with the use of (E.17)

$$\begin{aligned}\eta^{2m+1}\overset{0}{I}_1 = {} & 2\sqrt{\alpha t}\eta^{2m}e^{-\frac{\eta^2}{4\alpha t}}\{\mathrm{erfc}(\frac{\xi-\overset{2}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})-\mathrm{erfc}(\frac{\xi-\overset{1}{\xi}}{\eta}\sqrt{\frac{4\alpha t}{\eta^2}})\}\\ & -\frac{1}{\sqrt{\pi}}\{(\xi-\overset{2}{\xi})\eta^{2m}\mathrm{E}_1(\frac{(\xi-\overset{2}{\xi})^2+\eta^2}{4\alpha t})-(\xi-\overset{1}{\xi})\eta^{2m}\mathrm{E}_1(\frac{(\xi-\overset{1}{\xi})^2+\eta^2}{4\alpha t})\}\\ & -2\eta^{2m+1}\int_{\sqrt{\frac{\eta}{4\alpha t}}}^{\infty}e^{-u^2}\{\mathrm{erfc}(u\frac{\xi-\overset{2}{\xi}}{\eta})-\mathrm{erfc}(u\frac{\xi-\overset{1}{\xi}}{\eta})\}\,du \qquad m\geq 0\end{aligned} \tag{E.66}$$

Using the limits (23.9), (22.13) and (21.22) it can be seen that (E.66) is equal to zero at infinity

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty} \eta^{2m+1}\overset{0}{I}_1 = 0 \qquad m \geq 0 \tag{E.67}$$

The equation (E.61) together with the limits (23.9), (E.65) and (E.67) gives

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty} \eta^{2m+1}\overset{1}{I}_1 = 0 \qquad m \geq 1 \tag{E.68}$$

and by repeating the process with (E.61)

$$\lim_{\sqrt{\xi^2+\eta^2}\to\infty} \eta^{2m+1}\overset{k}{I}_1 = 0 \qquad m \geq k \tag{E.69}$$

Substitution of (E.69) in (E.59) shows that the condition (E.58) at infinity is fulfilled.

appendix F. list of symbols

roman symbols

A	matrix
a	constant
B	column matrix
b	constant
$\mathcal{C}$	constant in potential for far-field [L^3/T]
E	distributed extraction [L/T]
g	acceleration of gravity [L/T^2]
H	thickness of aquifer [L]
h	saturated thickness of aquifer [L]
$\bar{h}$	average saturated thickness of aquifer [L]
J_j	Bessel function of first kind of order j [-]
k	hydraulic conductivity [L/T]
L	length [L]
m	integer constant [-]
n	integer constant [-]
Q	discharge of continuous well [L^3/T]
$\underset{i}{Q}$	discharge of instantaneous well [L^3]
Q_j	discharge in direction j [L^2/T]
q_j	specific discharge in direction j [L/T]
$\mathcal{Q}$	discharge at infinity [L^3/T]
R	length [L]
r	radial coordinate [L]
S	storativity [-]
s_i	strength of instantaneous doublet [L^4]
t	time [T]
u	variable of integration
v_j	seepage velocity in direction j [L/T]
X	column matrix
x	cartesian horizontal coordinate [L]
y	cartesian horizontal coordinate [L]
z	cartesian vertical coordinate [L]

greek symbols

α	diffusivity [L^2/T]
β	factor for zero of Bessel function [1/L]
γ	Euler's constant [-]
	=.5772 15664 90153 28606 06512 (Abramowitz and Stegun, 1972)
ϵ	extraction of continuous areasink [L/T]
$\underset{0}{\epsilon}$	extraction of areasink of degree 0 [L/T]
$\underset{0j}{\epsilon}$	part linear in j of extraction of areasink of degree 0 [1/T]
ε	variation
η	local coordinate perpendicular to line element [L]
Θ	influence function for element with unknown strength
θ	angle [-]
Λ	influence function for element with known strength
λ	strength of continuous linedoublet [L^3/T]
$\underset{i}{\lambda}$	strength of instantaneous linedoublet [L^3]
ν	porosity [-]
Ξ	strength of ring-source [L^3/T]
ξ	local coordinate along line-element [L]
ρ	density groundwater [M/L^3]
σ	discharge of linesink [L^2/T]
$\underset{i}{\sigma}$	discharge of instantaneous linesink [L^2]
τ	variable of integration in time [T]
Φ	discharge potential [L^3/T]
φ	piezometric head [L]
Ω	complex potential ($\Re(\Omega) = \Phi$) [L^3/T]

subscripts

as	area-sink
c	constant in space
d	point-doublet
db	line-doublet
e	evapo-transpiration
f	final steady state
g	Glover(1974)'s solution
sh	line-segment with specified head
i	instantaneous
l	linear in space
ls	line-sink
n	degree n
o	initial steady state
p	far-field function for constant
r	far-field function for discharge at infinity
w	well
0	degree 0
1	degree 1

appendix G. list of figures